

# Reciprocal Reinsurance Treaties Under an Optimal and Fair Joint Survival Probability

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## Abstract

We study reinsurance treaties between an insurer and a reinsurer considering both parties' interests. Most articles only focus on the insurer's point of view. The latest research considering both sides have considerably oversimplified the joint survival function. This situation leads to an unrealistic optimal solution; one of the parties can make risk-free profits while the other bears all the risk. In this paper, we define and optimize a fair joint survival probability for a reciprocal reinsurance treaty, under the expected value principle, for both quota-share and stop-loss reinsurance contracts.

**Keywords:** Reinsurance Treaties, Optimization, Joint Survival, Expected Value Principle.

## 1 Introduction

Let us consider an insurer ( $I$ ) and a reinsurer ( $R$ ) entering a reinsurance treaty related to a risk ( $X$ ) for a fixed period of time. To be realistic and interesting for both actors, such a treaty should avoid any situation where one of the parties would cover the entire claim, while the other actor can make risk-free profits. Therefore, it is necessary to define the fair joint survival function ( $\Phi$ ) such that each party avoids bankruptcy and that the benefits of each party are  $\epsilon$ -comparable, where  $\epsilon$  establishes how fair the benefit is between the insurer and reinsurer. The properties for such a fair joint survival function will be detailed.

Several criteria can be selected to optimize a reinsurance treaty. Bowers (1997) and Vadja (1962) use the Variance measure. Cai and Tan (2007), Cai et al. (2008) and Lu et al. (2013) use the VaR and CTE risk measures. Also, Tan et al. use the CTE risk measure to minimize the insurer's total risk. In Kaluzka (2008), the expected utility is used, and in Arrow (1963) the expected concave utility function is considered. In Balbás et al. (2009), the authors analyze several risk functions such as the standard deviation, the absolute deviation and the conditional value at risk (CVaR). Chi and Tan (2011), Chi (2012) and Chi and Tan (2013) also consider the CVaR as well as the VaR measures, respectively under the expected value principle, variance related premium principles and general premium principles. Zhu (2013) use the Haezendonck risk measure to minimize the risk of the insurer. Cheung et al. (2014) consider the CTE and VaR measures as well as law-invariant convex risk measures. Cui et al. (2013) and Assa (2015) look at the distortion risk measure under respectively general premium principle, including the expected value and Wang's premium principle and distortion risk premium principle. All these studies consist in looking at the insurer's (or sometime to the reinsurer's as in Vadja (1962)

and Hürlimann (2011) ) point of view only. Using this perspective implies that a reinsurance treaty can be optimal for an actor (generally the insurer) while being unfavorable for the other one. Borsh (1960) is the first to consider the interests of both parties, and in Borch (1969) he suggested that a reinsurance contract might be optimal for the insurer without being acceptable for the reinsurer.

In Cai et al. (2013), a reciprocal approach is presented, providing retention amounts, based on the joint survival or profitable probability for several reinsurance treaties. The authors recognize that their results sometimes lead to unfair situations (p.158). This is precisely why we use a refined objective function, related to the joint survival and profitability functions that we optimize in this article. This approach allows us to avoid these unfair cases. We aim at providing a fair share of benefits between the insurer and reinsurer. We will focus on two types of contract; quota-share and the stop-loss models, using the expected value principle for the reinsurance premiums. Fang and Qu (2012) also consider these two types of contract under the same principle. In Castañer and Bielsa (2014) only the stop-loss reinsurance contract is considered. Fang and Qu (2012) and Castañer and Bielsa (2014) aim at maximizing the joint survival function in regards to both actors' interests. In our research, we go further by introducing a modified objective function which combine the survival of both parties with a fair share of benefits, for each type of contract. Balbás et al. (2013) use deviation measures and coherent risk measures for developing risk sharing strategies. However, the authors acknowledge that under particular conditions, the selected risk sharing plan might provoke a high probability of global bankruptcy (p. 55). Here, we develop an objective function that we optimize, ensuring that each party avoids bankruptcy.

Let us denote by  $P_I$  and  $P_R$  the net insurance premiums received by the insurer and the reinsurer, respectively. Also, let  $I_f$  and  $f$  represent the retained loss function (i.e. the loss covered by the insurer) and the ceded loss function (i.e. the reinsurer's part covered), respectively. Finally,  $u_I$  and  $u_R$  represent the initial wealths of the insurer and the reinsurer, respectively. We have that  $I_f(X) + f(X) = X$ , where  $X$  represents the loss. When both parties avoid bankruptcy, then  $I_f(X) \leq P_I + u_I$  and  $f(X) \leq P_R + u_R$ . Also, the benefits of each party being  $\epsilon$ -comparable is equivalent to  $P_I - I_f \leq \epsilon(P_R - f)$ , which can be interpreted as a way to fairly share the benefits between the insurer and reinsurer. We seek the highest probability so that both the insurer and the reinsurer avoid bankruptcy while at the same time share  $\epsilon$ -fairly the benefits. In this paper, we look at both actors' interests and we develop a new objective function,  $\Phi$ , the fair joint survival function.

In Section 2, we expand the equations leading to the characterization of the optimization problem under the expected value principle. In Section 3, we evaluate both  $\Phi_\epsilon^*$  and  $f^*$ , representing the optimal fair joint survival function and the optimal ceded loss function, respectively, considering the quota-share reinsurance model. In Section 4, we analyze the properties of the stop-loss reinsurance contract applied to this fair joint survival function. Section 5 concludes.

## 2 Optimal Fair Joint Survival Function Under The Expected Value Principle

In this section, we develop a fair joint survival function ( $\Phi$ ) that guarantees to avoid bankruptcy for both the insurer and the reinsurer, while sharing fairly the benefits from this transaction.

To ensure such conditions, we introduce a parameter ( $\epsilon$ ) that determines how fair the contract is between the insurer and the reinsurer. This parameter may vary depending on the type of the contract. We define the fair joint survival function by

$$\Phi_\epsilon(f) = Pr\{P_I - I_f \leq \epsilon(P_R - f), I_f \leq P_I + u_I, f \leq P_R + u_R\}.$$

Therefore, we seek to solve  $\max_{f \in \mathfrak{F}} \Phi_\epsilon(f)$  where  $\mathfrak{F}$  is the set of eligible functions  $f$  defining the contract between the insurer and the reinsurer.

Then, our objective is to define the optimal contract for each type of reinsurance model. We characterize the optimal solution ( $f^*$ ) and the optimal value ( $\Phi^*$ ) of the fair joint survival function. This function can be interpreted as an objective function and a probability function, resulting in a distribution function.

In this paper, we chose to consider the expected value principle for the reinsurance premiums, applied to two types of contract: the quota-share and the stop-loss models. Under this principle, the premiums can be formulated as:

$$P_I = (1 + \theta_I)E[I_f(X)] \text{ and } P_R = (1 + \theta_R)E[f(X)], \quad (1)$$

where  $P_I$  and  $P_R$  are respectively the insurer and the reinsurer net premium,  $\theta_R > 0$  and  $\theta_I > 0$  their relative safety loadings. For a matter of simplification, we avoid the case where  $\theta_I = \theta_R$ .

In the following section, we study the quota-share case.

### 3 Optimization of the Fair Joint Survival Function for a quota-share Reinsurance Model

Quota-share reinsurance is a common contract model. The insurer cedes an agreed-on percentage of the risk it insures. Let  $(1 - b)$  be this percentage, where  $b \in [0, 1]$ . Therefore, the retained loss function is given by

$$I_f(X) = bX, \quad (2)$$

and the ceded loss function is given by

$$f(X) = (1 - b)X. \quad (3)$$

Replacing (2) and (3) in (1) leads respectively to

$$P_I = (1 + \theta_I)b\mu \text{ and } P_R = (1 + \theta_R)(1 - b)\mu, \quad (4)$$

where  $\mu$  is the expected value of the loss  $X$ .

#### 3.1 Optimizing Algorithm for the Quota-Share model

The optimization problem we are studying is represented by

$$\max_{b \in [0,1]} \Phi_\epsilon(b) = Pr\{P_I - I_f \leq \epsilon(P_R - f), I_f \leq P_I + u_I, f \leq P_R + u_R\},$$

where  $\Phi_\epsilon(b)$  is the fair joint survival function. Furthermore, we have that  $u_I \geq 0$  and  $u_R \geq 0$ . The optimization problem is equivalent to the maximization, in terms of  $b$ , of the probability of

$$I_f \leq P_I + u_I, \quad f \leq P_R + u_R, \quad \text{and} \quad P_I - I_f \leq \epsilon(P_R - f). \quad (5)$$

Using equations (2), (3) and (4), we see that minimizing (5) is equivalent to minimizing the following inequalities, in terms of  $b$ ,

$$bx \leq (1 + \theta_I)b\mu + u_I, \quad (1 - b)x \leq (1 + \theta_R)(1 - b)\mu + u_R, \quad \text{and} \quad x \leq \frac{b\gamma_I - \epsilon\gamma_R(1 - b)}{b + b\epsilon - \epsilon}, \quad (6)$$

where  $\gamma_R = (1 + \theta_R)\mu$  and  $\gamma_I = (1 + \theta_I)\mu$ .

Therefore,  $X$  must obey three equations having  $b$  as unknown paramater. Having  $X$  being less than or equal to three limits is equivalent to  $X$  being less than or equal to the minimum of these limits. Let us define the limits to be respected by the loss  $X$  to ensure respectively the survival of the insurer ( $L_I$ ), the survival of the reinsurer ( $L_R$ ) and a fair share of the benefit ( $L_O$ ) as

$$L_I = \gamma_I + \frac{u_I}{b}, \quad L_R = \gamma_R + \frac{u_R}{1 - b}, \quad \text{and} \quad L_O = \frac{b\gamma_I - \epsilon\gamma_R(1 - b)}{b + b\epsilon - \epsilon}.$$

**Remark.** We set  $\epsilon > \frac{b}{1-b}$  to avoid problematic cases. If  $\epsilon \leq \frac{b}{1-b}$ , then the third equation in (6) would become

$$x \geq \frac{b\gamma_I - \epsilon\gamma_R(1 - b)}{b + b\epsilon - \epsilon},$$

which implies that  $\Phi^*$  would be equal to  $F(\min(L_I, L_R)) - F(L_O)$ . This situation results in substantially low values of  $\Phi^*$ , which are undesirable situations that both the insurer and reinsurer would avoid.

The next propositions compare the three limits to find the minimum ranges of  $b$  values that would optimize the fair joint survival function. In each comparison, two cases exist, whether  $\gamma = \gamma_I - \gamma_R$  is positive or negative. This option, which is equivalent to comparing  $(\theta_I)$  and  $(\theta_R)$  will determine the optimal value of the fair joint survival function. Figure 1 illustrates that this optimal value will be expressed in terms of the distribution function  $F(x)$ .

The next proposition compares  $L_I$  and  $L_R$ .

**Proposition 1.** Let  $\beta_{IR} = u_R + u_I - \gamma$  and  $\Delta_{IR} = \beta_{IR}^2 + 4\gamma u_I > 0$ . Then using  $b_{IR_1} = \frac{-\beta_{IR} - \sqrt{\Delta_{IR}}}{2\gamma}$  and  $b_{IR_2} = \frac{-\beta_{IR} + \sqrt{\Delta_{IR}}}{2\gamma}$  leads to two different cases. The first one concerns  $\gamma > 0$ . If  $b \in [b_{IR_1}, b_{IR_2}]$ , we have that  $L_R > L_I$ . On the other hand, when  $\gamma < 0$ , the comparison between  $L_I$  and  $L_R$  depends on the sign of  $\Delta_{IR}$  and whether  $b$  belongs or not to  $[b_{IR_1}, b_{IR_2}]$ . Thus,  $L_I > L_R$  if  $\Delta_{IR} > 0$  and  $b \in [b_{IR_1}, b_{IR_2}]$ .

Note that  $b \in [b_{IR_1}, b_{IR_2}]$  does not ensure  $L_R$  being greater than  $L_I$ . The comparison between the safety loadings of the insurer and the reinsurer matters as much as the range of  $b$ . Consequently, the choice of  $\theta_I$  and  $\theta_R$  and the initial wealths  $u_I$  and  $u_R$  will determine which actor imposes the limit to be respected by the loss  $X$  so that both of them avoid bankruptcy.

The next proposition compares  $L_I$  and  $L_O$ .

**Proposition 2.** Let  $\Delta_{IO} = \beta_{IO}^2 + 4\epsilon^2\gamma u_I > 0$ , where  $\beta_{IO} = u_I + \epsilon u_I - \epsilon\gamma$ . Then, using  $b_{IO_1} = \frac{-\beta_{IO} - \sqrt{\Delta_{IO}}}{2\epsilon\gamma}$  and  $b_{IO_2} = \frac{-\beta_{IO} + \sqrt{\Delta_{IO}}}{2\epsilon\gamma}$ , we establish that when  $\gamma > 0$ ,  $L_O > L_I$  is equivalent to  $b \in [b_{IO_1}, b_{IO_2}]$ .

However, when  $\gamma < 0$ ,  $L_I > L_O$  if and only if  $b \in [b_{IO_1}, b_{IO_2}]$  and  $\Delta_{IO} > 0$ .

Proposition 2 suggests, again, that the lower limit between  $L_I$  ensuring the survival of the insurer, and  $L_O$  ensuring a fair share of the benefits, depends on the value of the safety loadings difference  $\theta_I - \theta_R$  and the initial wealths of the insurer and reinsurer. Once these values are set, we can determine which values  $b$  can take depending on the comparison required between  $L_I$  and  $L_O$ .

The next proposition compares  $L_R$  and  $L_O$ .

**Proposition 3.** *Let  $\Delta_{RO} = \beta_{RO}^2 + 4\epsilon\gamma u_R > 0$ , where  $\beta_{RO} = u_R + \epsilon u_R - \gamma$ . Then, using  $b_{RO_1} = \frac{-\beta_{RO} - \sqrt{\Delta_{RO}}}{2\gamma}$  and  $b_{RO_2} = \frac{-\beta_{RO} + \sqrt{\Delta_{RO}}}{2\gamma}$ , we establish that when  $\gamma > 0$ ,  $L_O > L_R$  is equivalent to  $b \in [b_{RO_1}, b_{RO_2}]$ .*

*Whereas when  $\gamma < 0$ ,  $L_R > L_O$  if and only if  $\Delta_{RO} > 0$  and  $b \in [b_{RO_1}, b_{RO_2}]$ .*

The comparison between these limits  $L_R$  and  $L_O$  has the same interpretation as for the comparison of  $L_I$  and  $L_O$ .  $L_R$  can be greater than  $L_O$  depending on the parameters of the problem (i.e.  $u_I$ ,  $u_R$ ,  $\theta_I$  and  $\theta_R$ ).

We can illustrate the resulting optimal probabilities, based on different retention level ranges. Twelve cases are derived from the initial choice of  $\gamma$  representing the safety loadings difference, and then on the sets of possible values of  $b$ . This means that the safety loadings of each actor directly impacts the optimal result of their fair joint survival probability, with a comparable profit. More precisely, the comparison between these safety loadings is the starting point for the initial branches of the tree summarizing the algorithm produced in this section.

To illustrate the previous propositions leading to the decision tree, we here provide a simple illustration.

**Example 1.** *Consider a loss  $X$  following a Compound Poisson distribution with average  $E[X] = 2000$ . We set the safety loadings  $\theta_I = 0.04$  and  $\theta_R = 0.02$ , the initial wealths  $u_I = 300$  and  $u_R = 1500$ , and  $\epsilon = 0.7$ . We set an hypothetical value to the retention  $b = 0.4$  in order to follow the paths in Figure 1.*

*Browsing the decision tree and noting that  $\gamma > 0$ ,  $b \notin [b_{IR_1}; b_{IR_2}] = [-44.17; 0.17]$  and  $b \in [b_{RO_1}; b_{RO_2}] = [-63.17; 0.42]$ , we conclude that  $\Phi_\epsilon^*(b) = F(L_R(b)) = 0.9990071$ .*

*For purpose of verification, we check that  $\epsilon > \frac{b}{1-b}$ ,  $F(L_R(b)) > F(L_I(b))$  and  $F(L_R(b)) > F(L_O(b))$ .*

*In Example 1, we set  $b$  to illustrate the algorithm and obtain  $\Phi^*(b) = 0.9990071$ . The following section provides more realistic scenarios, optimizing the retention level to obtain maximal fair joint function values.*

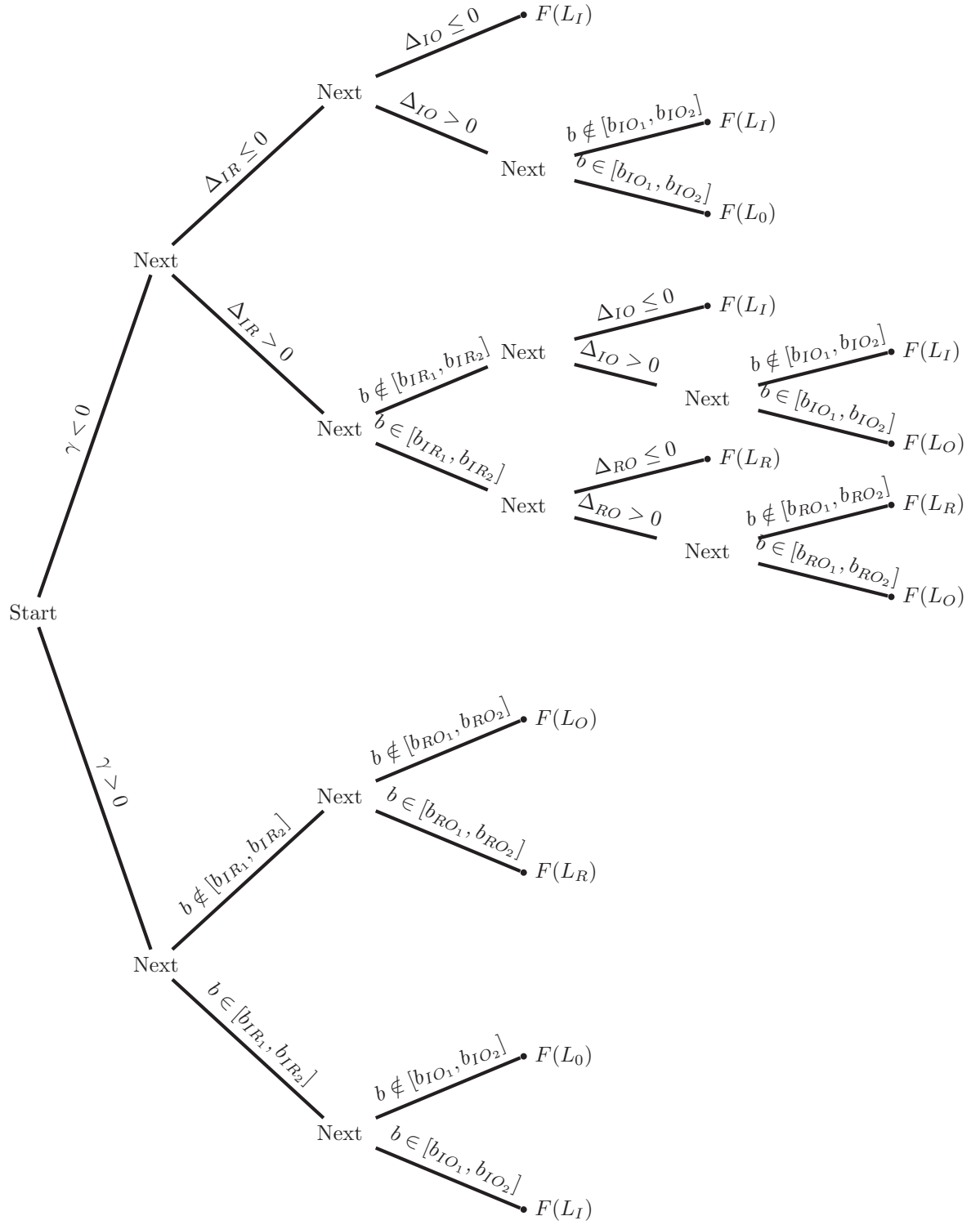


Figure 1: Decision tree for optimizing the fair joint survival function

### 3.2 Optimal Value of the Quota-Share Retention

In this section, we discuss how we can evaluate the optimal quota-share retention  $b^*$  such that  $f^*(x) = (1 - b^*)x$  defines the quota-share reinsurance contract that optimizes the fair joint survival function between the insurer and the reinsurer.

First, the optimal value  $b^*$  depends on the distribution function  $F(x)$ . Then, several properties can be summarized in the following proposition.

**Proposition 4.** *The maximization of  $\Phi_\epsilon(b)$  depends on the minimum of  $L_I, L_R$  and  $L_O$ . Maximizing  $F(L_I(b))$  is equivalent to minimizing  $b$  on the subset where  $L_I = \min(L_I, L_R, L_O)$ . Also, maximizing  $F(L_R(b))$  is equivalent to maximizing  $b$  on the subset where  $L_R = \min(L_I, L_R, L_O)$ . Finally, maximizing  $F(L_O(b))$  is equivalent to minimizing  $b$  if  $\gamma > 0$  and maximizing  $b$  if  $\gamma < 0$ , on the subset where  $L_O = \min(L_I, L_R, L_O)$ .*

Therefore, by referring to Figure 1, we can establish the following theorems.

**Theorem 1.** *For  $\gamma > 0$ , i.e. when the safety loading of the insurer ( $\theta_I$ ) is greater than the reinsurer's ( $\theta_R$ ), the optimal value of the fair joint survival function and of the quota-share retention are respectively*

$$\Phi_\epsilon^* = \max\{F(L_I(\sigma_1)); F(L_O(\sigma_2)); F(L_R(-\sigma_3)); F(L_O(\sigma_4))\}$$

and

$$b^* = \arg \max_{\sigma_i} \{F(L_I(\sigma_1)); F(L_O(\sigma_2)); F(L_R(-\sigma_3)); F(L_O(\sigma_4))\},$$

where  $\forall i \in [1; 4]$ ,  $\sigma_i = \min(\Sigma_i)$  and

- $\Sigma_1 = [0; 1] \cap [b_{IR1}; b_{IR2}] \cap [b_{IO1}; b_{IO2}]$ ,
- $\Sigma_2 = [0; 1] \cap [b_{IR1}; b_{IR2}] \setminus [b_{IO1}; b_{IO2}]$ ,
- $\Sigma_3 = -\{[0; 1] \cap [b_{RO1}; b_{RO2}] \setminus [b_{IR1}; b_{IR2}]\}$ ,
- $\Sigma_4 = [0; 1] \setminus \{[b_{IR1}; b_{IR2}] \cup [b_{RO1}; b_{RO2}]\}$ .

**Example 2.** *We illustrate Theorem 1 using the framework provided in Example 1. We can compare the results given by the optimization Theorem 1 and the results obtained from arbitrarily chosen values. Here, we want to obtain  $b$ , representing the unknown retention level. According to Theorem 1, we first need to evaluate the sets  $\Sigma_i$  and then determine  $\sigma_i = \min(\Sigma_i)$ . A quick computation leads to the following values,*

$\Sigma_i$		$\sigma_i$	
$\Sigma_1$	[0; 0.1698]	$\sigma_1$	0
$\Sigma_2$	$\emptyset$	$\sigma_2$	$\emptyset$
$\Sigma_3$	[-0.4156; -0.1698]	$\sigma_3$	-0.4156
$\Sigma_4$	[0.4156; 1]	$\sigma_4$	0.4156

*The next step is to evaluate  $F(L_I(\sigma_1))$ ,  $F(L_O(\sigma_2))$ ,  $F(L_R(-\sigma_3))$  and  $F(L_O(\sigma_4))$ . We obtain:  $F(L_R(-\sigma_3)) = 0.9991195$  as a maximum value. Finally, we conclude that the optimal retention value  $b^* = -\sigma_3 = 0.4156$  and  $\Phi_\epsilon^* = F(L_R(-\sigma_3)) = 0.999195$ .*

For negative values of  $\gamma$ , we refer to the upper half of Figure 1. We will use the symbol  $S \cap UC_{\{condition^*\}}$  to denote the subset  $S$  under *condition\**. If *condition\** is not respected, then  $S \cap UC_{\{condition^*\}} = \emptyset$ .

**Theorem 2.** For  $\gamma < 0$ , i.e. when the safety loading of the insurer ( $\theta_I$ ) is lower than the reinsurer's ( $\theta_R$ ), the optimal value of the fair joint survival function and of the quota-share retention are respectively

$$\Phi_\epsilon^* = \max\{F(L_O(-\pi_1)); F(L_R(-\pi_2)); F(L_R(-\pi_3)); F(L_O(-\pi_4)); F(L_I(\pi_5)); \\ F(L_I(\pi_6)); F(L_O(-\pi_7)); F(L_I(\pi_8)); F(L_I(\pi_9))\}$$

and

$$b^* = \arg \max_{\pi_i} \{F(L_O(-\pi_1)); F(L_R(-\pi_2)); F(L_R(-\pi_3)); F(L_O(-\pi_4)); F(L_I(\pi_5)); \\ F(L_I(\pi_6)); F(L_O(-\pi_7)); F(L_I(\pi_8)); F(L_I(\pi_9))\},$$

where  $\forall i \in [1; 9]$ ,  $\pi_i = \min(\Pi_i)$  such that:

- $\Pi_1 = -\{[0; 1] \cap [b_{IR1}; b_{IR2}] \cap [b_{RO1}; b_{RO2}]\} \cap UC_{\{\Delta_{IR}>0; \Delta_{RO}>0\}}$ ,
- $\Pi_2 = -\{[0; 1] \cap [b_{IR1}; b_{IR2}] \setminus [b_{RO1}; b_{RO2}]\} \cap UC_{\{\Delta_{IR}>0; \Delta_{RO}>0\}}$ ,
- $\Pi_3 = -\{[0; 1] \cap [b_{IR1}; b_{IR2}]\} \cap UC_{\{\Delta_{IR}>0; \Delta_{RO}\leq 0\}}$ ,
- $\Pi_4 = -\{[0; 1] \cap [b_{IO1}; b_{IO2}] \setminus [b_{IR}; b_{IR2}]\} \cap UC_{\{\Delta_{IR}>0; \Delta_{IO}>0\}}$ ,
- $\Pi_5 = \{[0; 1] \setminus \{[b_{IR1}; b_{IR2}] \cup [b_{IO1}; b_{IO2}]\}\} \cap UC_{\{\Delta_{IR}>0; \Delta_{IO}>0\}}$ ,
- $\Pi_6 = \{[0; 1] \setminus [b_{IR1}; b_{IR2}]\} \cap UC_{\{\Delta_{IR}>0; \Delta_{IO}>0\}}$ ,
- $\Pi_7 = -\{[0; 1] \cap [b_{IO1}; b_{IO2}]\} \cap UC_{\{\Delta_{IR}\leq 0; \Delta_{IO}>0\}}$ ,
- $\Pi_8 = \{[0; 1] \setminus [b_{IO1}; b_{IO2}]\} \cap UC_{\{\Delta_{IR}\leq 0; \Delta_{IO}>0\}}$ ,
- $\Pi_9 = [0; 1] \cap UC_{\{\Delta_{IR}\leq 0; \Delta_{IO}\leq 0\}}$ .

**Example 3.** To illustrate Theorem 2, we assume that  $X$  follows a Pareto distribution with parameters  $x_m = 1$  and  $k = 2$ . We set  $\theta_I = 0.02$ ,  $\theta_R = 0.04$ ,  $u_I = 5$ ,  $u_R = 10$  and  $\epsilon = 1.2$ . We obtain the following intermediary results,

Variables	Value	$\Pi_i$		$\pi_i$	
$\gamma$	-0.04	$\Pi_1$	$[-0.54; -1]$	$\pi_1$	-1
$\Delta_{IR}$	225.4	$\Pi_2$	$[-0.33; -0.54]$	$\pi_2$	-0.54
$\Delta_{IO}$	120.9	$\Pi_3$	$\emptyset$	$\pi_3$	$\emptyset$
$\Delta_{RO}$	483.8	$\Pi_4$	$\emptyset$	$\pi_4$	$\emptyset$
$[b_{IR1}; b_{IR2}]$	$[0.33; 375.7]$	$\Pi_5$	$[0; 0.33]$	$\pi_5$	0
$[b_{IO1}; b_{IO2}]$	$[0.54; 229.6]$	$\Pi_6$	$[0; 0.33]$	$\pi_6$	0
$[b_{RO1}; b_{RO2}]$	$[0.54; 550.5]$	$\Pi_7$	$\emptyset$	$\pi_7$	$\emptyset$
		$\Pi_8$	$\emptyset$	$\pi_8$	$\emptyset$
		$\Pi_9$	$\emptyset$	$\pi_9$	$\emptyset$

Consequently,  $F(L_O(-\pi_1)) = 0.7597078$  and  $F(L_R(-\pi_2)) = 0.9982374$  must be compared. We find that  $b^* = -\Pi_2 = 0.54$  is the optimal retention value and  $\Phi_\epsilon^* = F(L_R(-\pi_2)) = 0.9982374$  is the optimal fair joint probability value.



Theorem 1 and Theorem 2 can easily be derived from the algorithm presented in Figure 1.

In this section, we highlighted the importance of the optimal retention and the optimal value of the fair joint survival function. This optimal value is defined by  $F(L_I(b^*))$ ,  $F(L_R(b^*))$  and  $F(L_O(b^*))$ . Consequently, one can define, for a limited set of values for  $\epsilon$ , a fair reciprocal treaty under a quota-share contract. We will now consider the stop-loss contract.

## 4 Optimization of the Fair Joint Survival Function for a Stop-Loss Reinsurance Model

The stop-loss treaty is a non-proportional type of contract where the ceded loss function is represented by  $f(X) = \max\{X - d; 0\}$  and the retained loss function is given by  $I_f(X) = \min\{X; d\}$ . Consequently, knowing that  $E[I_f(X)] = E[X - f(X)]$ , (1) can be respectively expressed by  $P_R(d) = (1 + \theta_R) \int_d^\infty S(x) dx$ , and  $P_I(d) = \gamma_I - \frac{1+\theta_I}{1+\theta_R} P_R(d)$ , where  $\gamma_I = (1 + \theta_I)\mu$  and  $S$  is the survival function. The optimization of the fair joint survival function applied to a stop-loss is presented in what follows.

### 4.1 Optimal retention and fair joint survival function

In this section, we provide the conditions under which a fair reciprocal treaty under a stop-loss contract type exists. We also evaluate the optimal retention of this treaty, and the optimal value of the fair joint survival function.

**Theorem 3.** *For a stop-loss contract with  $d \geq 0$ , consider the following assumptions:*

- **H1:**  $d + (1 + \theta_I) \int_d^\infty S(x) dx - \gamma_I - u_I = 0$  has a unique solution  $\tilde{d}$ ,
- **H2:**  $d + (1 + \theta_I) \int_d^\infty S(x) dx - \gamma_I - \epsilon u_R = 0$  has a unique solution  $\hat{d}$ ,
- **H3:**  $\epsilon < \frac{u_I}{u_R}$ ,
- **H4:**  $\frac{\epsilon \theta_R + \theta_I}{\epsilon + 1} \leq \frac{F(\hat{d})}{1 - F(\hat{d})}$ , where  $\hat{d} = \min\{\tilde{d}, \hat{d}\}$ .

If **H1-H4** hold, then the optimal retention  $d^*$  and the optimal fair joint survival function  $\Phi_\epsilon^* = \Phi_\epsilon(d^*)$  exist and are defined as:

$$\begin{cases} d^* = \tilde{d} \text{ and } \Phi_\epsilon^* = F(P_R(\tilde{d}) + u_R + \tilde{d}) & \text{if } \theta_R \leq \frac{F(\hat{d})}{1 - F(\hat{d})}, \\ d^* = \hat{d} \text{ and } \Phi_\epsilon^* = F(P_R(\hat{d}) + u_R + \hat{d}) & \text{if } \theta_R > \frac{F(\hat{d})}{1 - F(\hat{d})}. \end{cases}$$

**Example 4.** *To illustrate Theorem 3, we assume that  $X$  follows an Exponential distribution with  $E[X] = 2$ . We set  $\theta_I = 0.02$ ,  $\theta_R = 0.04$ ,  $u_I = 10$ ,  $u_R = 5$  and  $\epsilon = 0.9$ , hence  $\gamma_I = 10.2$ . Because  $\int_d^\infty S(x) dx = \frac{\exp(-\lambda d)}{\lambda}$ ,  $d \gg (1 + \theta_I) \int_d^\infty S(x) dx$  when  $d \geq 5$ . Then,  $\tilde{d} = \gamma_I - u_I = 12.04$  is the unique solution of equation **H1**. Similarly,  $\hat{d} = \gamma_I - \epsilon u_R = 6.54$  is the unique solution of equation **H2**. As  $\hat{d} = \min\{\tilde{d}, \hat{d}\} = 6.54$ , then  $F(\hat{d}) = 0.9619936$  and thus hypotheses **H3** and **H4** hold. Finally,  $\theta_R \leq \frac{F(\hat{d})}{1 - F(\hat{d})}$ . Then the optimal retention is  $d^* = \tilde{d} = 12.04$  and the optimal fair joint probability is  $\Phi_\epsilon^* = F(P_R(\tilde{d}) + u_R + \tilde{d}) = 0.9975703$ .*

Under certain hypotheses, we can define a fair reciprocal treaty between the insurer and the reinsurer based on a stop-loss contract. The optimal value of the retention and the optimal value of the fair joint survival function depend on  $\epsilon$ , the variable defining the agreement between the two actors. This variable itself depends, for its admissible bound, on the initial wealths ( $u_I$  and  $u_R$ ) of the insurer and reinsurer. The larger  $\frac{u_I}{u_R}$  is, the more flexibility on  $\epsilon$  has.

## 5 Conclusion

In this paper, we introduce a new method to obtain a balanced joint survival and fair joint profitability between an insurer and a reinsurer for quota-share and stop-loss reinsurance treaties, under the expected value principle. Our motivation is to develop such a reciprocal reinsurance treaty that would be at the same time in the best interest of both stakeholders and optimal. Usually, research that focus on these reinsurance treaties only consider the insurer's point of view, leading to unacceptable situation for the reinsurer: situations where the reinsurer can become bankrupt while the insurer makes all the benefits. Here, we use a fairness variable ( $\epsilon$ ), which can vary depending on the actors' agreement, and under certain conditions. We restricted our study to contracts under the expected value principle. Other models than quota-share and stop-loss could be considered, and other pricing principles.

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# A Proofs

## A.1 Proof of Proposition 1

Comparing  $L_I$  and  $L_R$ , we have that

$$L_I - L_R = 0 \Leftrightarrow \gamma b^2 + (u_R + u_I - \gamma)b - u_I = 0, \quad (7)$$

where  $\gamma = \gamma_I - \gamma_R$ . If  $\gamma > 0 \Leftrightarrow \theta_I > \theta_R$ , then (7) can take three schemes, as illustrated in Figure 2. Figure 2 shows that a curve can have zero, one or two roots (i.e. respectively be over, reach on one unique point, or cross the  $X$ -axis).

Moreover, because  $\Delta_{IR} = \beta_{IR}^2 + 4\gamma u_I > 0$ , where  $\beta_{IR} = u_R + u_I - \gamma$ , only the lowest curve describes this case.

Then, using  $b_{IR_1} = \frac{-\beta_{IR} - \sqrt{\Delta_{IR}}}{2\gamma}$  and  $b_{IR_2} = \frac{-\beta_{IR} + \sqrt{\Delta_{IR}}}{2\gamma}$ , we obtain that if  $b \in [b_{IR_1}, b_{IR_2}]$ , then  $L_R > L_I$ . Otherwise,  $L_I > L_R$ .

In the second case, when  $\gamma < 0 \Leftrightarrow \theta_R > \theta_I$ , the quadratic function (7) can take three schemes, as illustrated in Figure 3.

Therefore, if  $\Delta_{IR} > 0$  and if  $b \in [b_{IR_1}, b_{IR_2}]$  then  $L_I > L_R$ .

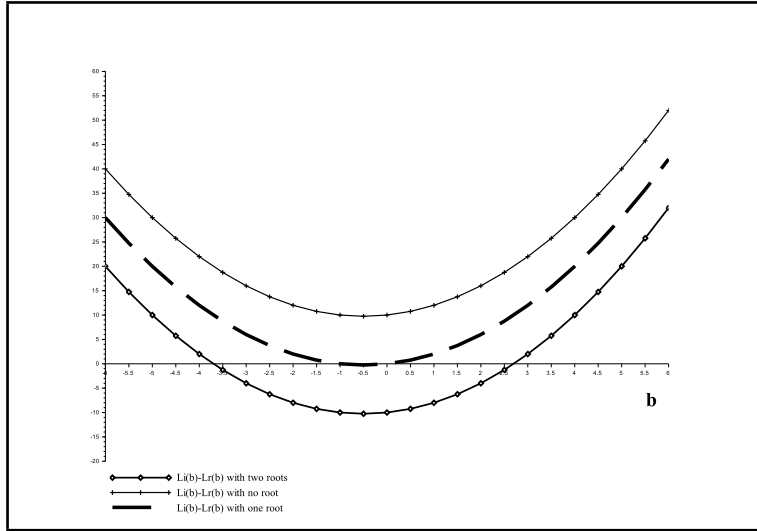


Figure 2: Three schemes of the quadratic function (7) when  $\gamma > 0$ .

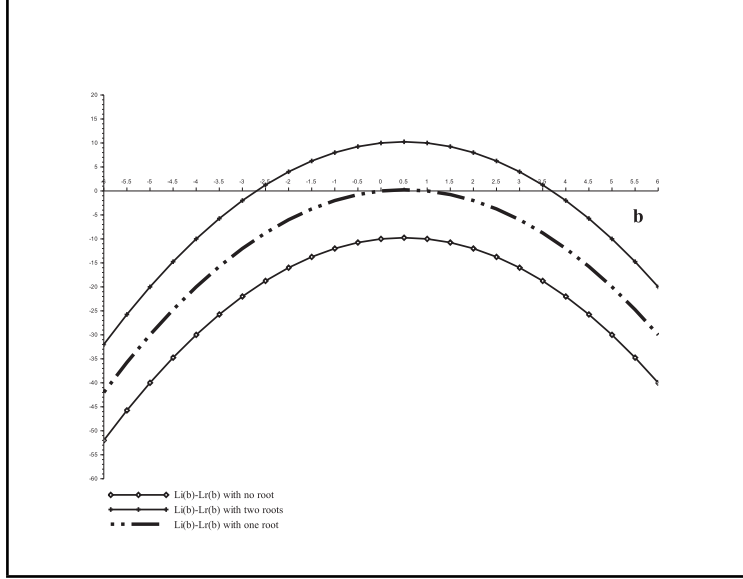


Figure 3: Three schemes of the quadratic function equation (7) when  $\gamma < 0$ .

## A.2 Proof of Proposition 2

We use the same reasoning as for the proof of Proposition 1 with

$$L_I - L_O = 0 \Leftrightarrow \gamma \epsilon b^2 + (u_I + \epsilon u_I - \epsilon \gamma)b - \epsilon u_I = 0.$$

## A.3 Proof of Proposition 3

Again, the same demonstration as for the proof of Proposition 1 holds considering the following equivalence:

$$L_R - L_O = 0 \Leftrightarrow \gamma b^2 + (u_R + \epsilon u_R + \gamma_R - \gamma_I)b - \epsilon u_R = 0.$$

## A.4 Proof of Proposition 4

We have that  $F(x)$  is increasing for  $x \in ]0; +\infty[$ ,  $L_I(b) = \gamma_I + \frac{u_I}{b}$  is decreasing for  $b \in [0; 1]$ ,  $L_R(b) = \gamma_R + \frac{u_R}{1-b}$  is increasing for  $b \in [0; 1]$ , and  $L_O(b) = \frac{(\gamma_I + \gamma_R)b - \epsilon \gamma_R}{(1 + \epsilon)b - \epsilon}$ , for  $b \in [0; 1]$ , is increasing if  $\gamma < 0$  (i.e. if  $\theta_I < \theta_R$ , respectively the insurer and reinsurer safety loadings); and decreasing if  $\gamma > 0$  (i.e.  $\theta_I > \theta_R$ ).

## A.5 Proof of Theorem 3

We have that  $f(X) = \max\{X - d, 0\}$ ,  $I_f(X) = \min\{X, d\}$ ,  $P_I = (1 + \theta_I)E[I_f(x)]$ ,  $P_R = (1 + \theta_R)E[f(x)]$  and  $d \in \mathbb{R}^+$ . To solve  $\max_{d \in \mathbb{R}^+} \Phi_\epsilon(d) = \Pr\{I_f \leq P_I + u_I; f \leq P_R + u_R; P_I - I_f \leq \epsilon(P_R - f)\}$ , where  $\Phi_\epsilon(d)$  is the fair joint survival function, one must maximize, in terms of  $d$ , the probability of

$$\begin{cases} I_f \leq P_I + u_I, \\ f \leq P_R + u_R, \\ P_I - I_f \leq \epsilon(P_R - f), \end{cases} = \begin{cases} \min\{X, d\} \leq P_I + u_I, \\ \max\{X - d, 0\} \leq P_R + u_R, \\ P_I - \min\{X, d\} \leq \epsilon(P_R - \max\{X - d, 0\}). \end{cases}$$

Therefore, the fair joint survival function can be expressed by

$$\begin{aligned}\Phi_\epsilon(d) &= Pr\{X \leq d\}Pr\{X \leq P_I + u_I, 0 \leq P_R + u_R, P_I - X \leq \epsilon P_R | X \leq d\} \\ &\quad + Pr\{X > d\}Pr\{d \leq P_I + u_I, X - d \leq P_R + u_R, P_I - d \leq \epsilon(P_R - X + d) | X > d\}.\end{aligned}$$

Consequently,

$$\Phi_\epsilon(d) = \begin{cases} Pr\{X \leq P_I + u_I, X \geq P_I - \epsilon P_R\} & \text{if } d > P_I + u_I, \\ Pr\{X \leq P_R - \frac{P_I}{\epsilon} + (1 + \frac{1}{\epsilon})d, X \leq d + P_R + u_R\} & \text{if } d \leq P_I + u_I. \end{cases} \quad (8)$$

To evaluate the second case in (8), we need to compare the two limits that  $X$  must respect. This is equivalent to

$$P_R - \frac{P_I}{\epsilon} + (1 + \frac{1}{\epsilon})d \geq d + P_R + u_R \Leftrightarrow d \geq P_I + \epsilon u_R.$$

Thus, the fair joint survival function is expressed by

$$\Phi_\epsilon(d) = \begin{cases} F(P_I + u_I) - F(P_I - \epsilon P_R) & \text{if } d > P_I + u_I, \\ F(P_R - \frac{P_I}{\epsilon} + (1 + \frac{1}{\epsilon})d) & \text{if } d \leq \min\{P_I + u_I, P_I + \epsilon u_R\}, \\ F(d + P_R + u_R) & \text{if } d \in [P_I + \epsilon u_R; P_I + u_I]. \end{cases} \quad (9)$$

The last case of equation (9) exists if and only if  $P_I + u_I > P_I + \epsilon u_R$ , which is equivalent to hypothesis **H1** ( $\epsilon < \frac{u_I}{u_R}$ ).

By considering (9),  $\Phi_\epsilon(d)$  can take three expressions ( $\Phi^1$ ,  $\Phi^2$  and  $\Phi^3$ ), depending on the values of  $d$  respectively on  $]P_I + u_I; +\infty[$ ,  $[0; \min\{P_I + u_I, P_I + \epsilon u_R\}]$  and  $[P_I + \epsilon u_R; P_I + u_I]$ . We have that

$$\Phi^1(d) = F(P_I(d) + u_I) - F(P_I(d) - \epsilon P_R(d)), \quad (10)$$

$$\Phi^2(d) = F(P_R(d) - \frac{P_I(d)}{\epsilon} + (1 + \frac{1}{\epsilon})d), \quad (11)$$

and

$$\Phi^3(d) = F(d + P_R(d) + u_R). \quad (12)$$

To maximize  $\Phi^1(d)$  defined by equation (10), we use

$$\begin{aligned}P_R(d) &= (1 + \theta_R)E[f(X)] \\ &= (1 + \theta_R)E[\max\{X - d; 0\}] \\ &= (1 + \theta_R) \int_d^\infty S(x) dx,\end{aligned}$$

and

$$\begin{aligned}P_I(d) &= (1 + \theta_I)E[I_f(X)] \\ &= (1 + \theta_I)E[X - f(X)] \\ &= (1 + \theta_I)\mu - \frac{1 + \theta_I}{1 + \theta_R}P_R(d) \\ &= \gamma_I - \frac{1 + \theta_I}{1 + \theta_R}P_R(d).\end{aligned}$$

With the differentiation formula for  $h(x) = \int_{a(x)}^{\infty} f(x, y) dy$ , that is given by

$$\frac{dh(x)}{dx} = \int_{a(x)}^{\infty} \frac{\partial f(x, y)}{\partial x} dy - f(x, a(x)) \frac{da(x)}{dx},$$

we obtain

$$P'_R(d) = -(1 + \theta_R)S(d) \quad (13)$$

and

$$P'_I(d) = (1 + \theta_I)S(d). \quad (14)$$

Hence,

$$[P_I(d) + u_I]' = (1 + \theta_I)S(d)$$

and

$$[P_I(d) - \epsilon P_R(d)]' = [(1 + \theta_R)\epsilon + (1 + \theta_I)]S(d).$$

We can conclude that  $[P_I(d) - \epsilon P_R(d)]' > [P_I(d) + u_I]'$  which means that the increase of  $[P_I(d) - \epsilon P_R(d)]$  is greater than the increase of  $[P_I(d) + u_I]$ . Since  $F(x)$  is an increasing function, the increase of  $F(P_I(d) - \epsilon P_R(d))$  is greater than the increase of  $F(P_I(d) + u_I)$ . Thus,  $\Phi^1(d)$  is decreasing, and because  $\Phi^1(d)$  is defined on the open ball  $]P_I + u_I; +\infty[$  it never reaches its maximum, i.e.  $\max \Phi^1(d)$  has no solution.

Secondly, to maximize  $\Phi^2(d)$  defined by (11) we use  $H(d) = P_R(d) - \frac{P_I(d)}{\epsilon} + (1 + \frac{1}{\epsilon})d$ . According to (13) and (14), we have that  $H(d)' = (1 + \theta_R + \frac{1 + \theta_I}{\epsilon})(F(d) - 1) + \frac{1 + \epsilon}{\epsilon}$ .

Therefore,

$$H(d)' \geq 0 \Leftrightarrow \frac{\epsilon \theta_R + \theta_I}{\epsilon + 1} \leq \frac{F(d)}{1 - F(d)}.$$

One can easily verify that  $\frac{F(d)}{1 - F(d)}$  is an increasing function of  $d$ ,  $\forall d \geq 0$ . Because  $\Phi^2(d)$  is defined for  $d \in [0; \hat{d}]$ , where  $\hat{d} = \min\{\hat{d}, \tilde{d}\}$ , then  $H(d)$  and  $F(H(d)) = \Phi^2(d)$  are increasing  $\forall d \in [0; \hat{d}]$  if and only if  $\frac{\epsilon \theta_R + \theta_I}{\epsilon + 1} \leq \frac{F(0)}{1 - F(0)}$ , which is impossible since  $F(0) = 0$  and  $\frac{\epsilon \theta_R + \theta_I}{\epsilon + 1} > 0$ .

Moreover  $H(d)$  and  $F(H(d)) = \Phi^2(d)$  are decreasing  $\forall d \in [0; \hat{d}]$  if and only if  $\frac{\epsilon \theta_R + \theta_I}{\epsilon + 1} \leq \frac{F(\hat{d})}{1 - F(\hat{d})}$ , which contradicts hypothesis **H4**. Therefore  $\Phi^2(d)$  has no optimum. Finally, to maximize  $\Phi^3(d)$  defined by (12), we use  $G(d) = P_R(d) + u_R + d$  and  $G(d)' = (1 + \theta_R)F(d) - \theta_R$ . Therefore,  $G(d)' \geq 0 \Leftrightarrow \theta_R \leq \frac{F(d)}{1 - F(d)}$ , where  $d \in [\hat{d}; \tilde{d}]$ . Because  $\frac{F(d)}{1 - F(d)}$  is an increasing function of  $d$ , we conclude that  $G(d)$  is increasing, and thus  $\Phi^3(d) = F(G(d))$  is increasing, on  $[\hat{d}; \tilde{d}]$ , if and only if  $\theta_R \leq \frac{F(\hat{d})}{1 - F(\hat{d})}$ ; then  $\Phi^*_\epsilon = \max \Phi^3(d) = F(P_R(\tilde{d}) + u_R + \tilde{d})$  and  $d^* = \tilde{d}$ . On the other hand,  $G(d)$  is decreasing, which means that  $\Phi^3(d) = F(G(d))$  is decreasing on  $[\hat{d}; \tilde{d}]$ , if and only if  $\theta_R > \frac{F(\tilde{d})}{1 - F(\tilde{d})}$ . Then,  $\Phi^*_\epsilon = \max \Phi^3(d) = F(P_R(\hat{d}) + u_R + \hat{d})$  and  $d^* = \hat{d}$ .