ONE-YEAR AND TOTAL RUN-OFF RESERVE RISK ESTIMATORS BASED ON HISTORICAL ULTIMATE ESTIMATES

ABSTRACT

In this contribution we present closed-form formulas in order to estimate, based on the historical triangle of estimated ultimates, both the one-year and the total run-off reserve risk. This is helpful in case (as often usual in practice) the reserve risk formulas related to the applied reserving methodology are unknown or in case such formulas can not be rigorously derived since a fully well defined stochastic model supporting the reserving methodology is missing (for example due to mixing of reserving methods).

KEYWORDS

Reserve risk, mean square error of prediction, one-year prediction uncertainty, total run-off prediction uncertainty, ultimate estimates.

1 INTRODUCTION

Within the stochastic claims reserving theory it is good practice to derive estimators for both the one-year and the total runoff reserve risk (prediction uncertainty) whenever a new reserving model is defined. This is usually done considering the appropriate conditional mean square error of prediction (MSEP).

For example for the standard Chain-Ladder (CL) methodology, which is the most popular reserving methodology and is generally supported by Mack's distribution-free model (see Mack [3]), the total run-off prediction uncertainty estimator is usually given by Mack's formula and the one-year prediction uncertainty estimator is given by Merz-Wüthrich's formula (see Merz-Wüthrich [5]).

Unfortunately not for all existing stochastic reserving models such estimators have been derived yet in a closed-form and for many reserving methodologies used in practice within the insurance industry a fully well defined supporting stochastic model is missing (often because of mixture of basics methods) and therefore, the reserve risk estimators can not be rigorously derived.

The lack of such formulas is a currently open problem in the insurance industry (see Dal Moro-Lo [1]). The central question is:

How to quantify the prediction uncertainties (both one-year and total run-off) if we are faced with the above stated situation?

In this paper we suggest an answer to this question. We will take the following unusual approach (as already proposed in Rehman-Klugman [7]):

We do not specify the stochastic reserving model supporting the reserving methodology (since, as already mentioned, this cannot always be well defined) but only assume a stochastic model for the ultimate estimates. That allows us to derive an estimator for the conditional MSEP of the prediction uncertainties (both one-year and total run-off) based on the historical triangle of ultimate estimates only, i.e. regardless of which underlying reserving methodology generates the ultimate estimates.

In the following we denote with $\widehat{U}_{i,j}$ the estimated ultimate claims amount for accident year (AY) $i \in \{0,\dots,I\}$ at development period (DP) $j \in \{0,\dots,I\}$:

i/j	0	1	...	$I-1$	I
0	$\widehat{U}_{0,0}$	$\widehat{U}_{0,1}$	...	$\widehat{U}_{0,I-1}$	$\widehat{U}_{0,I}$
1	$\widehat{U}_{1,0}$	$\widehat{U}_{1,1}$	...	$\widehat{U}_{1,I-1}$	
$\vdots$	$\vdots$	$\vdots$			
$I-1$	$\widehat{U}_{I-1,0}$	$\widehat{U}_{I-1,1}$			
I	$\widehat{U}_{I,0}$				

Basically, our stochastic model for the ultimate estimates assumes that, for any accident year i, the estimated ultimate losses of two consecutive development periods j and $j+1$ are related by a proportionality factor g_j which only depends on the development period j, i.e. it holds true

$$\widehat{U}_{i,j+1} \approx g_j \cdot \widehat{U}_{i,j}.$$

This assumption is very similar to what the CL reserving method assumes for claims data triangles (cumulative payments or incurred losses) and in the context of our model, the factors (g_j) can be expected to be close to 1, since the relationship is established between estimated ultimate losses.

Based on the available ultimate estimates data, the main result for the one-year reserve risk will be:

$$\begin{aligned}\widehat{\text{msep}}_{\sum_{i=0}^{I}\widehat{\text{CDR}}_i(I+1)|\mathscr{F}_I}(0) = &\sum_{i=1}^{I}\left[\widehat{\sigma}_{I-i}^2 \cdot \widehat{U}_{i,I-i} + (\widehat{g}_{I-i}-1)^2 \cdot \widehat{U}_{i,I-i}^2\right] \\ &+ 2\sum_{1\le i<j\le I}(\widehat{g}_{I-i}\cdot\widehat{g}_{I-j} - \widehat{g}_{I-i} - \widehat{g}_{I-j} + 1)\cdot \widehat{U}_{i,I-i}\cdot\widehat{U}_{j,I-j}.\end{aligned} \tag{1.1}$$

where, for a specific development period j, the estimated development factor $\widehat{g}_j$ is a weighted average of the individual accident year development factors $\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}}$, while $\widehat{\sigma}_j^2$ is a measure of the volatility of these development factors around their weighted average in absolute terms, i.e. scaled to the estimated ultimate amounts.

Moreover, based on the available ultimate estimates data, the main result for the total run-off reserve risk will be:

$$\begin{aligned}&\widehat{\text{msep}}_{\sum_{i=0}^{I} U^i|\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^{I} U^i \middle| \mathscr{F}_I\right]\right) \\ &= \sum_{i=1}^{I}\left\{\left[\sum_{k=I-i}^{I-1}\left(\prod_{j=I-i}^{k-1}\widehat{g}_j\right)\cdot\widehat{\sigma}_k^2\cdot\left(\prod_{l=k+1}^{I-1}\widehat{g}_l^2\right)\right]\cdot\widehat{U}_{i,I-i} + \left(1-\prod_{j=I-i}^{I-1}\widehat{g}_j\right)^2\cdot\widehat{U}_{i,I-i}^2\right\} \\ &\quad + 2\sum_{1\le i<j\le I}\left(1-\prod_{k=I-i}^{I-1}\widehat{g}_k\right)\left(1-\prod_{k=I-j}^{I-1}\widehat{g}_k\right)\cdot\widehat{U}_{i,I-i}\cdot\widehat{U}_{j,I-j}.\end{aligned} \tag{1.2}$$

Remarks:

- Both estimators (1.1) and (1.2) are given by the sum of two components: the first component is the sum of all risk estimators for single accident years (first row), the second component is an additional "covariance" term (second row) as it generally appears when calculating the variance of the sum of random variables.
  Moreover, the first component of each estimator consists of the sum of two terms representing the "process variance" (first term) and the "parameter estimation error", respectively (second term).

- Note that both estimators (1.1) and (1.2) can be easily implemented in a spreadsheet and therefore they have an high potential for application in actuarial practice.
- Within our model the following implication holds true:
  No parameter uncertainty in the ultimate estimates generating process implies the proportionality factors (g_j) to be equal to 1.
  Note that this is reflected within our results since in case of no underlying parameter uncertainty the estimated factors $(\widehat{g}_j)$ will be forced to be equal to 1, and as a consequence the "parameter estimation error" and the "covariance" terms in (1.1) and (1.2) are equal to 0.

The paper will be organized as follows:

In section 2 we provide some technical notations and definitions.
In section 3 we precisely formulate our model assumptions for the estimated ultimates, which allows us to derive unbiased estimators for the model parameters without assuming independence between accident years.
In section 4 we derive the mean square error of prediction for the one-year reserve risk within our model.
In section 5 we derive the mean square error of prediction of the total run-off reserve risk within our model.
In section 6 we compare our formulas with Mack's (total run-off view) and Merz-Wüthrich's (one-year view) formulas and provide some toy numerical examples for didactic purposes.
In the appendix the technically oriented readers can find all the rigorous details for deriving the presented formulas.

In order to facilitate the practically oriented readers, throughout the paper we will evaluate step by step the following numerical example for which we already show here the main results.
We consider the following triangle of estimated ultimates $(\widehat{U}_{i,j})$

i/j	0	1	2	3	4	5	6	7	8	9	10	11	12
0	233'927	225'734	224'101	223'193	222'383	223'724	223'743	223'765	223'669	223'601	223'558	223'558	223'558
1	230'176	224'714	222'707	221'871	225'215	223'977	224'027	223'933	223'907	224'004	224'036	224'036	
2	225'361	219'714	217'321	223'774	225'325	223'809	223'680	223'450	223'544	223'660	223'697		
3	211'284	215'963	218'664	217'849	215'981	214'561	214'391	214'395	214'468	214'445			
4	212'024	221'439	220'313	218'938	218'873	220'610	220'938	221'576	221'628				
5	289'863	298'147	295'716	293'527	293'758	294'427	294'713	294'673					
6	268'568	273'222	272'219	272'657	271'057	269'817	269'763						
7	261'945	275'360	274'935	274'262	273'275	271'810							
8	232'400	241'510	242'330	247'851	247'245								
9	235'521	251'022	267'514	267'747									
10	239'385	246'401	245'398										
11	257'904	259'549											
12	262'936												

The estimated parameters $(\widehat{g}_j)$ and $(\widehat{\sigma}_j^2)$ (see also section 3.1 for more insights) are given by

$$\widehat{g}_j = \frac{\sum_{i=0}^{I-j-1} \widehat{U}_{i,j+1}}{\sum_{i=0}^{I-j-1} \widehat{U}_{i,j}},$$

$$\widehat{\sigma}_j^2 = \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} \widehat{U}_{i,j} \left(\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}} - \widehat{g}_j \right)^2,$$

and can be easily evaluated to be equal to

j	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{g}_j$	1.0188	1.0030	1.0024	0.9996	0.9984	1.0002	1.0002	1.0001	1.0001	1.0000	1.0000	1.0000
$\widehat{\sigma}_j^2$	241.45	118.62	37.85	11.80	8.32	0.16	0.41	0.03	0.04	0.01	0.00	0.00

The square rooted estimators (1.1) and (1.2) as well as Mack's and Merz-Wüthrich's results are reported in the below table

	One-year risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I} \widehat{\text{CDR}}_i(I+1)\mid\mathscr{F}_I}(0)$	Total run-off risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I} U^i\mid\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^{I} U^i \Big\mid \mathscr{F}_I\right]\right)$
our Model	12'025	15'228
underlying CL Model	11'203 (Mack)	13'457 (Merz-Wüthrich)

Note that our formulas deliver similar results as Mack's and Merz-Wüthrich's formulas and that the latter have been evaluated using the related claims payments triangle $(C_{i,j})$ given by

i/j	0	1	2	3	4	5	6	7	8	9	10	11	12
0	157'495	212'770	219'680	220'978	221'276	223'724	223'743	223'765	223'669	223'601	223'558	223'558	223'558
1	154'969	213'352	219'201	220'469	222'751	223'958	224'005	224'030	223'975	224'048	224'036	224'036	
2	152'833	209'969	214'692	220'040	223'467	223'754	223'752	223'593	223'585	223'688	223'697		
3	144'223	207'644	212'443	214'108	214'661	214'610	214'564	214'484	214'459	214'459			
4	145'612	209'604	214'161	215'982	217'962	220'783	221'078	221'614	221'616				
5	196'695	282'621	288'676	290'036	292'206	294'531	294'671	294'705					
6	181'381	260'308	266'497	269'130	269'404	269'691	269'720						
7	177'168	263'130	268'848	270'787	271'624	271'688							
8	156'505	230'607	237'102	244'847	245'940								
9	157'839	239'723	261'213	264'755									
10	159'429	233'309	239'800										
11	169'990	246'019											
12	173'377												

2 NOTATIONS AND DEFINITIONS

Consider accident years $i \in \{0, \dots, I\}$ and development periods $j \in \{0, \dots, I\}$.
Denote with U^i the ultimate claim amount (random variable) for accident year i.
Denote with $\mathscr{F}_k$, $k \in \{0, \dots, 2I\}$, the (unspecified) total information available to the insurance company at the end of calendar year k (as usual we consider here the run-off situation).
Denote with $\widehat{U}_{i,j}$ the estimated ultimate claims amount for AY i at DP j, i.e. the estimated ultimate claims amount for accident year i at the end of calendar year $i+j$.

Define the following sets of information:

$$\mathscr{D}^I := \sigma\left(\bigcup_{i=0}^{I} \{\widehat{U}_{i,0}, \dots, \widehat{U}_{i,I-i}\}\right) \subseteq \mathscr{F}_I,$$

$$\mathscr{D}^{I+1} := \sigma\left(\bigcup_{i=0}^{I} \{\widehat{U}_{i,0}, \dots, \widehat{U}_{i,\min(I-i+1,I)}\}\right) \subseteq \mathscr{F}_{I+1},$$

$$\mathscr{D}_j^{j+k} := \sigma\left(\bigcup_{i=0}^{I} \{\widehat{U}_{i,0}, \dots, \widehat{U}_{i,\max(j+k-i,j)}\}\right), \quad k, j \in \{0, \dots, I\}.$$

Remarks:

- As already mentioned, the total information available $\mathscr{F}_k$, $k \in \{0, \dots, 2I\}$, is unspecified, i.e. it could be generated by the claims payments or by the claims incurred amounts (or even both) and also include additional information.
- The information $\mathscr{D}^k$, $k \in \{0, \dots, 2I\}$, is a subset of the total information $\mathscr{F}_k$ and contains the available information given by the ultimates estimates only, which means that we do not focus anymore on the underlying claims payments or claims incurred amounts.
- The information $\mathscr{D}_j^{j+k}$, $k, j \in \{0, \dots, I\}$, contains the ultimates estimates information at the end of calendar year $j+k$ as well as the ultimates estimates information until development period j for all accident years (as introduced in Dahms [2]).

3 THE MODEL ASSUMPTIONS

As outlined in the introduction we will not specify the stochastic reserving model supporting the reserving methodology but only

(A) assume the underlying reserving methodology is generating ultimate estimates,

(B) assume a stochastic model for the ultimate estimates.

In mathematical terms we make the following model assumptions:

Model Assumptions 1 (Estimated ultimates model)**.**

(A)
- $U^i = \widehat{U}_{i,I}, \quad i \in \{0, \dots, I\}$, *i.e. at development period I the ultimate estimates are fully developed*
- $\widehat{U}_{i,j}$ *is $\mathscr{F}_{i+j}$-measurable,* $\quad i, j \in \{0, \dots, I\}$
- *The 'best estimate' ultimate $E[U^i|\mathscr{F}_{i+j}]$, $i, j \in \{0, \dots, I\}$, can be expressed as a function of $\mathscr{F}_{i+j}$-measurable random variables and a collection of underlying parameters*
- $\widehat{U}_{i,j} = \widehat{E}[U^i|\mathscr{F}_{i+j}]$, $i, j \in \{0, \dots, I\}$, *is an estimated ultimate position obtained replacing the unknown underlying parameters with appropriately $\mathscr{F}_{i+j}$-measurable parameter estimators*

(B) There exist parameters $g_0, \dots, g_{I-1}$ and $\sigma_0^2, \dots, \sigma_{I-1}^2$ such that for all $i \in \{0, \dots, I\}$ and $j \in \{0, \dots, I-1\}$

$$E[\widehat{U}_{i,j+1}|\mathscr{D}_j^{j+i}] = g_j \cdot \widehat{U}_{i,j},$$

$$Var(\widehat{U}_{i,j+1}|\mathscr{D}_j^{j+i}) = \sigma_j^2 \cdot \widehat{U}_{i,j}.$$

Remarks:

- Assumption *(A)* expresses the traditional ultimate estimates representation in claims reserving (see equations 2.9 and 2.10 in Merz-Wüthrich [9]) and allows to also model the situation where a fully well defined stochastic model supporting the reserving methodology is missing.
- Note that we do not require any quality assumptions with respect to the underlying parameter estimators. This allows for more flexibility, since ad-hoc estimators can be considered as well (which is very common in practice, see for instance the concluding remarks in Merz-Wüthrich [9] page 390).
- Assumption *(B)* is very similar to the "Linear Stochastic Reserving Method" assumptions in Dahms [2] and note that no independence assumption between accident years is stated.
- The above mentioned underlying parameters are the ones that need to be estimated when generating the ultimate estimates $\widehat{U}_{i,j}$.
- Note that considering reserving methodologies for which the unconditional unbiased property related to the ultimate estimates is fulfilled (for instance in case of no underlying parameter uncertainty), i.e. for which it holds true

  $$E[\widehat{U}_{i,j}] = E[U^i], \quad i, j \in \{0, \dots, I\},$$

  our model assumptions implies

  $$E[U^i] = E[\widehat{U}_{i,j+1}] = E[E[\widehat{U}_{i,j+1}|\mathscr{D}_j^{j+i}]] = E[g_j \cdot \widehat{U}_{i,j}] = g_j \cdot E[U^i].$$

  As a consequence, the parameters (g_j) result to be equal to one and therefore they do not need to be estimated (see the remarks in section 3.1).
  In this respect please note that, in practice, the above unconditional unbiased property is mostly approximately fulfilled but not exactly fulfilled, even when considering well established reserving methodologies like the traditional Bornhuetter-Ferguson (BF, see [9],[4],[8]) method (see equation 2.15 in Merz-Wüthrich [9]) or the Generalized Linear Models (GLM) (see remark 6.15 in Merz-Wüthrich [9]). Therefore, the parameters (g_j) in our model are generally close to one, but not necessary equal to one.

3.1 PARAMETER ESTIMATION

The parameters (g_j) and (σ_j^2) can be estimated by the following $\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\ldots,\widehat{U}_{i,j}\}\right)$-conditionally unbiased estimators

$$\widehat{g}_j := \frac{\sum_{i=0}^{I-j-1}\widehat{U}_{i,j+1}}{\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}}, \quad j \in \{0,\ldots,I-1\}, \tag{3.1}$$

$$\widehat{\sigma}_j^2 := \frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}\left(\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}} - \widehat{g}_j\right)^2, \quad j \in \{0,\ldots,I-2\}. \tag{3.2}$$

For $\widehat{\sigma}_{I-1}^2$ we use the estimator proposed by Mack [3] given by

$$\widehat{\sigma}_{I-1}^2 = \min\left\{\widehat{\sigma}_{I-3}^2, \widehat{\sigma}_{I-2}^2, \frac{\widehat{\sigma}_{I-2}^4}{\widehat{\sigma}_{I-3}^2}\right\}. \tag{3.3}$$

Proof. See appendix A.1. □

Remarks:

- In case the unconditional unbiased property related to the ultimate estimates is fulfilled, the parameters (g_j) do not need to be estimated and we set

  $$\widehat{g}_j = 1, \quad j \in \{0,\ldots,I-1\}.$$

  Moreover the parameter estimator for σ_j^2 needs to be slightly modified to

  $$\widehat{\sigma}_j^2 = \frac{1}{I-j}\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}\left(\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}} - 1\right)^2, \quad j \in \{0,\ldots,I-2\}.$$

- The estimated parameters $(\widehat{g}_j)$ and $(\widehat{\sigma}_j^2)$ related to our numerical example are given by

j	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{g}_j$	1.0188	1.0030	1.0024	0.9996	0.9984	1.0002	1.0002	1.0001	1.0001	1.0000	1.0000	1.0000
$\widehat{\sigma}_j^2$	241.45	118.62	37.85	11.80	8.32	0.16	0.41	0.03	0.04	0.01	0.00	0.00

- Note that our theory can be easily extended for considering the case where the accident years are not necessarily fully developed at the end of development period I. In this case tail parameters $\widehat{g}_I$ and $\widehat{\sigma}_I^2$ are required (for example obtained by additional extrapolation) and the sums in our results (1.1) and (1.2) must be extended by the additional terms where $i = 0$ or $j = 0$.

3.2 SOME BASIC RESULTS

In this section we provide some basic results that can be derived from our Model Assumptions 1. These will play a central role in sections 4 and 5 when deriving the estimators for the prediction uncertainties.

The following relationships hold true:

$$E[\widehat{U}_{i,I-i+1}|\mathscr{D}^I] = \widehat{U}_{i,I-i}\cdot g_{I-i}, \quad i \geq 1, \tag{3.4}$$

$$E[\widehat{U}_{i,I}|\mathscr{D}^I] = \widehat{U}_{i,I-i}\cdot\prod_{j=I-i}^{I-1} g_j, \quad i \geq 1, \tag{3.5}$$

$$E[\widehat{U}_{i,I}|\mathscr{D}^{I+1}] = \widehat{U}_{i,I-i+1} \cdot \prod_{j=I-i+1}^{I-1} g_j, \quad i \geq 2. \tag{3.6}$$

Furthermore, since $\widehat{U}_{i,I-i}$ is $\mathscr{D}^I$-measurable, it also holds true:

$$\begin{aligned} Var(\widehat{U}_{i,I-i+1}|\mathscr{D}^I) &= E[Var(\widehat{U}_{i,I-i+1}|\mathscr{D}_{I-i}^{I-i+i})|\mathscr{D}^I] + Var(E[\widehat{U}_{i,I-i+1}|\mathscr{D}_{I-i}^{I-i+i}]|\mathscr{D}^I) \\ &= E[\sigma_{I-i}^2 \cdot \widehat{U}_{i,I-i}|\mathscr{D}^I] + Var(g_{I-i} \cdot \widehat{U}_{i,I-i}|\mathscr{D}^I) \\ &= \sigma_{I-i}^2 \cdot E[\widehat{U}_{i,I-i}|\mathscr{D}^I] + g_{I-i}^2 \cdot \underbrace{Var(\widehat{U}_{i,I-i}|\mathscr{D}^I)}_{=0} \\ &= \sigma_{I-i}^2 \cdot \widehat{U}_{i,I-i}, \quad i \geq 1. \end{aligned} \tag{3.7}$$

Remark:
Please note that in the following sections we will make use of the following notation: $\widehat{E}[\cdot]$ (respectively $\widehat{Var}(\cdot)$) will denote the estimator for $E[\cdot]$ (respectively $Var(\cdot)$) which is obtained by final replacement of the unknown parameters $(g_j), (\sigma_j^2)$ with their parameter estimators $(\widehat{g}_j), (\widehat{\sigma}_j^2)$ after having performed ordinary computations.

3.3 THREE EXAMPLES

In order to clarify Model Assumptions 1. in this section we provide three examples of methodologies for generating ultimate estimates. The first one is the standard CL methodology, the second one is a modification of the latter, which better reflects what is generally done in actuarial practice, when parameters are adjusted according to expert judgment. The third is a credibility mixture between CL and BF.
The readers familiar with the above methodologies may skip the following subsections and directly go to the concluding remark 3.4.

3.3.1 STANDARD CHAIN LADDER METHODOLOGY

We denote with $C_{i,j}$ the cumulative payments for accident year $i \in \{0, \ldots, I\}$ up to development period $j \in \{0, \ldots, I\}$ and with $\mathscr{F}_k$, $k \in \{0, \ldots, 2I\}$, the total information available to the insurance company at the end of calendar year k which is in this case given by

$$\mathscr{F}_k := \sigma\left(\{C_{i,j}\}_{i+j \leq k}\right).$$

Under the standard CL framework the underlying parameters are given by a collection of factors (f_j) and the quantities $E[U^i|\mathscr{F}_{i+j}]$ respectively $\widehat{U}_{i,j}$ are defined as

$$E[U^i|\mathscr{F}_{i+j}] := C_{i,j} \cdot \prod_{l=j}^{I-1} f_l, \quad i, j \in \{0, \ldots, I\},$$

$$\widehat{U}_{i,j} := C_{i,j} \cdot \prod_{l=j}^{I-1} \widehat{f}_l^{(i+j)}, \quad i, j \in \{0, \ldots, I\}, \tag{3.8}$$

where the estimated CL factors at the end of calendar year k are given by

$$\widehat{f}_j^{(k)} := \begin{cases} \frac{\sum_{i=0}^{\min(I,k-j-1)} C_{i,j+1}}{\sum_{i=0}^{\min(I,k-j-1)} C_{i,j}}, & k-j \geq 1 \\ \bar{f}_j, & k-j < 1 \end{cases}, \quad k \in \{0, \ldots, 2I\}, \tag{3.9}$$

and $\bar{f}_0, \ldots, \bar{f}_{I-1}$ are known prior expected CL factors.

Remarks:

- The above choice of the estimated CL factors can be justified by Mack's distribution-free model assumptions (see [3]).
- Note that in practice the standard CL methodology is not often rigorously applied, since the estimated CL factors are often modified and/or smoothed according to actuarial judgment. This fact can be interpreted as one does not perfectly believe in Mack's distribution-free model.
- Note that assuming our Model Assumptions 1. with respect to the standard CL methodology (as defined above) implies that the cumulative payments process $(C_{i,j})$ fulfills conditions which do not comply with Mack's distribution-free model assumptions. As a consequence we can expect that the uncertainty estimators derived from our model do not necessary coincide with Mack's and Merz-Wüthrich's formulas.
  The implied conditions related to the cumulative payments process $(C_{i,j})$ are similar but more complex than in Mack's distribution-free model.

3.3.2 MODIFIED CHAIN LADDER METHODOLOGY

In this section we define a modified CL methodology which better reflects common actuarial practice.
We consider the underlying filtration given by

$$\mathscr{F}_k := \sigma\left(\{C_{i,j}, \varepsilon_{i,j}\}_{i+j\leq k}\right),$$

where the variables $\varepsilon_{i,j}$ are intended to model an adjustment to the standard CL estimator in order to take into account additional information not included in the information generated by the claims payments.
The underlying parameters are given by a collection of factors (f_j) and the quantities $E[U^i|\mathscr{F}_{i+j}]$ and $\widehat{U}_{i,j}$ respectively, are defined as

$$E[U^i|\mathscr{F}_{i+j}] := \varepsilon_{i,j} \cdot C_{i,j} \cdot \prod_{l=j}^{I-1} f_l, \quad i,j \in \{0,\ldots,I\},$$

$$\widehat{U}_{i,j} := \varepsilon_{i,j} \cdot C_{i,j} \cdot \prod_{l=j}^{I-1} \widetilde{f}_l^{(i+j)}, \quad i,j \in \{0,\ldots,I\}, \tag{3.10}$$

where $\left(\widetilde{f}_l^{(i+j)}\right)$ are appropriate $\mathscr{F}_{i+j}$-measurable estimators for (f_l).

Remarks:

- The above estimated CL factors $\left(\widetilde{f}_l^{(k)}\right)$ at the end of calendar year k could, for instance, coincide with the, in this case ad-hoc, standard CL estimators $\left(\widehat{f}_l^{(k)}\right)$.
- Compared to the standard CL methodology, the modified CL methodology better reflects the usual actuarial behaviour observed in practice, where the standard CL ultimate estimators/factors are possibly adjusted 'by hand' using expert opinion in order to take into account additional information such as change of legal practice, high inflation, job market information, etc.

3.3.3 CREDIBILITY MIXTURE BETWEEN CL AND BF

In this section we define a credibility mixture methodology between CL and BF following the setup of section 4.2 in Merz-Wüthrich [9]. We consider the underlying filtration given by

$$\mathscr{F}_k := \sigma\left(\{C_{i,j}\}_{i+j\leq k}, \{\widehat{\mu}_i\}_{i\leq k}\right),$$

where $\widehat{\mu}_i$ is an unbiased estimators for $E[C_{i,I}]$ which is independent of $C_{i,I-i}$ and $C_{i,I}$.

The underlying parameters are given by a collection of factors $(f_j), (\mu_i), (c_i)$ and the quantities $E[U^i|\mathscr{F}_{i+j}]$ and $\widehat{U}_{i,j}$ respectively, are defined as

$$E[U^i|\mathscr{F}_{i+j}] := c_i \cdot \underbrace{C_{i,j} \cdot \prod_{l=j}^{I-1} f_l}_{\text{CL component}} + (1-c_i) \cdot \underbrace{\left(C_{i,j} + \left(1 - \frac{1}{\prod_{l=j}^{I-1} f_l}\right)\mu_i\right)}_{\text{BF component}}, \quad i,j \in \{0,\ldots,I\},$$

$$\widehat{U}_{i,j} := \widehat{c}_i \cdot C_{i,j} \cdot \prod_{l=j}^{I-1} \widehat{f}_l^{(i+j)} + (1-\widehat{c}_i) \cdot \left(C_{i,j} + \left(1 - \frac{1}{\prod_{l=j}^{I-1} \widehat{f}_l^{(i+j)}}\right)\widehat{\mu}_i\right), \quad i,j \in \{0,\ldots,I\},$$

where $\widehat{c}_i$ is an appropriate estimator for c_i, for instance the one obtained by minimizing the appropriate unconditional MSEP for known parameters (f_j).

Remark:
The above estimated ultimates $(\widehat{U}_{i,j})$ are motivated by the underlying stochastic model given by assumption 4.11 in Merz-Wüthrich [9] and note that the standard CL estimators $\left(\widehat{f}_l^{(k)}\right)$ are ad-hoc parameter estimators within that model.

3.4 CONCLUDING REMARK

Finishing this section we would like to recall and highlight again that our methodology has been appositely designed for modeling the situation where a fully well defined stochastic model supporting the reserving methodology is missing (for example due to mixing of reserving methods or when underlying parameter estimators are adjusted 'by hand', as usually done in practice). In this case we can namely assume our Model Assumptions 1. without any limitation.

Therefore, if a stochastic reserving model supporting a reserving methodology can be fully well defined, as for the standard CL methodology where all the underlying parameter estimators can be perfectly motivated, and model validation delivers good results, then we believe it would be better to use (if known) the uncertainty estimators derived within the underlying model itself and not follow our theory.

However, since underlying stochastic reserving models

- do often rely on strong assumptions,
- the unbiasedness property of the parameter estimators cannot always be exactly proved,
- the prediction uncertainty estimators are sometimes derived using approximations,

one should in this case not forget to also allow for a (model) risk loading.
As a consequence, since the latter is generally not easily quantified, we consider our theory and the related "high-level" formulas to be in any case a very valuable alternative.

4 ONE-YEAR PREDICTION UNCERTAINTY

In this section we concentrate on the one-year reserve risk. Our goal is to derive the estimator (1.1) for the mean square error of prediction of the one-year prediction uncertainty. As usual in claim reserving we first focus on the result for a single accident year and then, in a second step, derive an estimator for the aggregated view.

Let us first define the observed claims development result $\widehat{\mathrm{CDR}}_i(I+1)$ as follows

$$\widehat{\mathrm{CDR}}_i(I+1) := \widehat{E}[U^i|\mathscr{F}_{I+1}] - \widehat{E}[U^i|\mathscr{F}_I], \quad i \geq 1.$$

4.1 SINGLE ACCIDENT YEAR

For a single accident year $i \geq 1$ the mean square error of prediction of the one-year prediction uncertainty is given by

$$\begin{aligned} \mathrm{msep}_{\widehat{\mathrm{CDR}}_i(I+1)|\mathscr{F}_I}(0) &:= E\left[\left(\widehat{E}[U^i|\mathscr{F}_{I+1}] - \widehat{E}[U^i|\mathscr{F}_I] - 0\right)^2 \middle| \mathscr{F}_I\right] \\ &= E\left[\left(\widehat{U}_{i,I-i+1} - \widehat{U}_{i,I-i}\right)^2 \middle| \mathscr{F}_I\right]. \end{aligned} \tag{4.1}$$

4.1.1 ONE-YEAR UNCERTAINTY ESTIMATOR

Making use of the relationships derived in section 3.2 we can derive the following estimator (see appendix B.1 for more details):

Estimator 1 (one-year reserve risk estimator for single accident year)**.** *Under Model Assumptions 1. we have the following estimator for the one-year prediction uncertainty for a single accident year*

$$\widehat{msep}_{\widehat{CDR}_i(I+1)|\mathscr{F}_I}(0) = \widehat{\sigma}^2_{I-i} \cdot \widehat{U}_{i,I-i} + (\widehat{g}_{I-i} - 1)^2 \cdot \widehat{U}^2_{i,I-i}, \quad i \geq 1. \tag{4.2}$$

Remarks:

- Note that, since

$$E\left[\left(\widehat{U}_{i,I-i+1} - \widehat{U}_{i,I-i}\right)^2 \middle| \mathscr{D}^I\right] = E\left[E\left[\left(\widehat{U}_{i,I-i+1} - \widehat{U}_{i,I-i}\right)^2 \middle| \mathscr{F}_I\right] \middle| \mathscr{D}^I\right], \quad i \geq 1,$$

  our one-year reserve risk estimator fulfills the following "unbiasedness" property

$$\widehat{\mathrm{msep}}_{\widehat{\mathrm{CDR}}_i(I+1)|\mathscr{F}_I}(0) = \widehat{E}\left[\mathrm{msep}_{\widehat{\mathrm{CDR}}_i(I+1)|\mathscr{F}_I}(0) \middle| \mathscr{D}^I\right], \quad i \geq 1. \tag{4.3}$$

- The one-year reserve risk estimators for single accident year related to our numerical example are shown in the following table

i	$\widehat{U}_{i,I-i}$	$\widehat{\text{msep}}_{\widehat{\text{CDR}}_i(I+1)\|\mathscr{F}_I}(0)$	$\widehat{\text{msep}}^{1/2}_{\widehat{\text{CDR}}_i(I+1)\|\mathscr{F}_I}(0)$
0	223'558	0	0
1	224'036	0	0
2	223'697	0	0
3	214'445	1'977	44
4	221'628	8'941	95
5	294'673	8'995	95
6	269'763	114'752	339
7	271'810	45'029	212
8	247'245	2'214'687	1'488
9	267'747	3'170'237	1'781
10	245'398	9'631'068	3'103
11	259'549	31'380'299	5'602
12	262'936	87'858'844	9'373

4.1.2 ARTIFICIAL SPLIT OF THE ONE-YEAR PREDICTION UNCERTAINTY BETWEEN PROCESS VARIANCE AND PARAMETER ESTIMATION ERROR

Note that the mean square error of prediction of the one-year prediction uncertainty can be split into three components (process variance, parameter estimation uncertainty and mixed term) as follows (see appendix B.2 for more details):

$$\begin{aligned}
\text{msep}_{\widehat{\text{CDR}}_i(I+1)|\mathscr{F}_I}(0) &:= E\left[\left(\widehat{E}[U^i|\mathscr{F}_{I+1}] - \widehat{E}[U^i|\mathscr{F}_I] - 0\right)^2 \middle| \mathscr{F}_I\right] \\
&= \underbrace{Var\left(E[U^i|\mathscr{F}_{I+1}]\middle|\mathscr{F}_I\right)}_{\text{process variance}} \\
&\quad + \underbrace{E\left[\left(\widehat{U}_{i,I-i+1} - E[U^i|\mathscr{F}_{I+1}]\right)^2 \middle| \mathscr{F}_I\right] + E\left[\left(\widehat{U}_{i,I-i} - E[U^i|\mathscr{F}_I]\right)^2 \middle| \mathscr{F}_I\right]}_{\text{parameter estimation uncertainty: first term}} \\
&\quad \underbrace{- 2E\left[\left(\widehat{U}_{i,I-i} - E[U^i|\mathscr{F}_I]\right)\left(\widehat{U}_{i,I-i+1} - E[U^i|\mathscr{F}_{I+1}]\right)\middle|\mathscr{F}_I\right]}_{\text{parameter estimation uncertainty: second term}} \qquad (4.4) \\
&\quad + \underbrace{2E\left[\left(\widehat{U}_{i,I-i+1} - E[U^i|\mathscr{F}_{I+1}]\right)\left(E[U^i|\mathscr{F}_{I+1}] - E[U^i|\mathscr{F}_I]\right)\middle|\mathscr{F}_I\right]}_{\text{mixed term}}.
\end{aligned}$$

In the appendix B.3, B.4 and B.5 we will additionally derive an estimator for each of the above terms.
As just shown we can not clearly split the one-year prediction uncertainty between process variance and parameter estimation uncertainty due to the presence of the mixed term.
Nevertheless we can at least provide the following artificial split of the one-year prediction uncertainty estimator

$$\widehat{\text{msep}}_{\widehat{\text{CDR}}_i(I+1)|\mathscr{F}_I}(0) = \underbrace{\widehat{\sigma}^2_{I-i} \cdot \widehat{U}_{i,I-i}}_{\text{"estimated process variance"}} + \underbrace{(\widehat{g}_{I-i} - 1)^2 \cdot \widehat{U}^2_{i,I-i}}_{\text{"estimated parameter estimation uncertainty"}}, \quad i \geq 1,$$

since for known underlying parameters (i.e. no underlying parameter uncertainty) the estimated parameters $\widehat{g}_j$, $j \in \{0, \ldots, I-1\}$, are forced to be equal to 1 and in consequence the above "estimated parameter estimation uncertainty" term is vanishing.

4.2 AGGREGATED ACCIDENT YEARS

In this section we take care of the aggregated accident year view. For aggregated accident years the mean square error of prediction of the one-year prediction uncertainty is given by

$$\begin{aligned}\mathrm{msep}_{\sum_{i=0}^{I}\widehat{\mathrm{CDR}}_i(I+1)|\mathscr{F}_I}(0) &:= E\left[\left(\sum_{i=0}^{I}\underbrace{\left(\widehat{E}[U^i|\mathscr{F}_{I+1}]-\widehat{E}[U^i|\mathscr{F}_I]\right)}_{=:\widehat{\mathrm{CDR}}_i(I+1)}-0\right)^2\middle|\mathscr{F}_I\right]\\ &= \sum_{i=1}^{I}E\left[\left(\widehat{E}[U^i|\mathscr{F}_{I+1}]-\widehat{E}[U^i|\mathscr{F}_I]\right)^2\middle|\mathscr{F}_I\right] \qquad (4.5)\\ &\quad+2\sum_{1\le i<j\le I}E\left[\left(\widehat{E}[U^i|\mathscr{F}_{I+1}]-\widehat{E}[U^i|\mathscr{F}_I]\right)\left(\widehat{E}[U^j|\mathscr{F}_{I+1}]-\widehat{E}[U^j|\mathscr{F}_I]\right)\middle|\mathscr{F}_I\right].\end{aligned}$$

For the above first term $\sum_{i=1}^{I}E\left[\left(\widehat{E}[U^i|\mathscr{F}_{I+1}]-\widehat{E}[U^i|\mathscr{F}_I]\right)^2\middle|\mathscr{F}_I\right]$ we already know an appropriate estimator, since it refers to the one-year prediction uncertainty for single accident years. Therefore it only remains to estimate the second term, which we will call one-year "covariance" term. This is done in the next subsection.

4.2.1 ONE-YEAR UNCERTAINTY ESTIMATOR (FOR AGGREGATED ACCIDENT YEARS)

We first state the following lemma:

Lemma 1. *Under Model Assumptions 1. and for $i<j$ it holds true*

$$Cov\left(\widehat{U}_{i,I-i+1},\widehat{U}_{j,I-j+1}\middle|\mathscr{D}^I\right)=0. \qquad (4.6)$$

Proof. See appendix B.6. □

Using Lemma 1. we obtain the following estimator for the one-year "covariance" term (see appendix B.7 for more details):

$$\begin{aligned}&2\sum_{1\le i<j\le I}\widehat{E}\left[\left(\widehat{U}_{i,I-i+1}-\widehat{U}_{i,I-i}\right)\left(\widehat{U}_{j,I-j+1}-\widehat{U}_{j,I-j}\right)\middle|\mathscr{D}^I\right] \qquad (4.7)\\ &\qquad=2\sum_{1\le i<j\le I}(\widehat{g}_{I-i}\cdot\widehat{g}_{I-j}-\widehat{g}_{I-i}-\widehat{g}_{I-j}+1)\cdot\widehat{U}_{i,I-i}\cdot\widehat{U}_{j,I-j},\quad i<j.\end{aligned}$$

Remark:
The one-year "covariance" term estimator related to our numerical example is given by

$$2\sum_{1\le i<j\le I}\widehat{E}\left[\left(\widehat{U}_{i,I-i+1}-\widehat{U}_{i,I-i}\right)\left(\widehat{U}_{j,I-j+1}-\widehat{U}_{j,I-j}\right)\middle|\mathscr{D}^I\right]=10'167'783.$$

As a consequence we can derive estimator (1.1):

Estimator 2 (one-year reserve risk estimator for aggregated accident years). *Under Model Assumptions 1. we have the following estimator for the one-year prediction uncertainty for aggregated accident years*

$$\widehat{msep}_{\sum_{i=0}^{I}\widehat{CDR}_i(I+1)|\mathscr{F}_I}(0) = \sum_{i=1}^{I}\left[\widehat{\sigma}_{I-i}^2 \cdot \widehat{U}_{i,I-i} + (\widehat{g}_{I-i}-1)^2 \cdot \widehat{U}_{i,I-i}^2\right] + 2\sum_{1\le i<j\le I}(\widehat{g}_{I-i}\cdot\widehat{g}_{I-j} - \widehat{g}_{I-i} - \widehat{g}_{I-j} + 1)\cdot\widehat{U}_{i,I-i}\cdot\widehat{U}_{j,I-j}.$$

Remarks:

- It is straightforward that the above estimator does fulfill the following "unbiasedness" property

$$\widehat{\mathrm{msep}}_{\sum_{i=0}^{I}\widehat{\mathrm{CDR}}_i(I+1)|\mathscr{F}_I}(0) = \widehat{E}\left[\mathrm{msep}_{\sum_{i=0}^{I}\widehat{\mathrm{CDR}}_i(I+1)|\mathscr{F}_I}(0)\middle|\mathscr{D}^I\right]. \tag{4.8}$$

- The one-year reserve risk estimator for aggregated accident years related to our numerical example is shown in the following table

i	$\widehat{U}_{i,I-i}$	$\widehat{\mathrm{msep}}_{\widehat{\mathrm{CDR}}_i(I+1)\mid\mathscr{F}_I}(0)$	$\widehat{\mathrm{msep}}^{1/2}_{\widehat{\mathrm{CDR}}_i(I+1)\mid\mathscr{F}_I}(0)$
0	223'558	0	0
1	224'036	0	0
2	223'697	0	0
3	214'445	1'977	44
4	221'628	8'941	95
5	294'673	8'995	95
6	269'763	114'752	339
7	271'810	45'029	212
8	247'245	2'214'687	1'488
9	267'747	3'170'237	1'781
10	245'398	9'631'068	3'103
11	259'549	31'380'299	5'602
12	262'936	87'858'844	9'373
"covariance" term		10'167'783	
	$\sum_{i=0}^{I}\widehat{U}_{i,I-i}$	$\widehat{\mathrm{msep}}_{\sum_{i=0}^{I}\widehat{\mathrm{CDR}}_i(I+1)\mid\mathscr{F}_I}(0)$	$\widehat{\mathrm{msep}}^{1/2}_{\sum_{i=0}^{I}\widehat{\mathrm{CDR}}_i(I+1)\mid\mathscr{F}_I}(0)$
Total	3'226'487	144'602'611	12'025

5 TOTAL RUN-OFF PREDICTION UNCERTAINTY

In this section we concentrate on the total run-off reserve risk. Our goal is to derive the estimator (1.2) for the mean square error of prediction of the total run-off prediction uncertainty. As usual we first focus on the result for a single accident year and then, in a second step, derive an estimator for the aggregated view.

5.1 SINGLE ACCIDENT YEAR

For a single accident year $i \geq 1$ the mean square error of prediction of the total run-off prediction uncertainty is given by

$$\begin{aligned}\text{msep}_{U^i|\mathscr{F}_I}(\widehat{E}[U^i|\mathscr{F}_I]) &:= E\left[\left(U^i - \widehat{E}[U^i|\mathscr{F}_I]\right)^2 \middle| \mathscr{F}_I\right] \\ &= E\left[\left(U^i - E[U^i|\mathscr{F}_I]\right)^2 \middle| \mathscr{F}_I\right] + E\left[\left(\widehat{E}[U^i|\mathscr{F}_I] - E[U^i|\mathscr{F}_I]\right)^2 \middle| \mathscr{F}_I\right] \\ &\quad + 2\underbrace{E\left[\left(U^i - E[U^i|\mathscr{F}_I]\right)\left(E[U^i|\mathscr{F}_I] - \widehat{E}[U^i|\mathscr{F}_I]\right) \middle| \mathscr{F}_I\right]}_{=0} \qquad (5.1) \\ &= \underbrace{E\left[\left(\widehat{U}_{i,I} - E[\widehat{U}_{i,I}|\mathscr{F}_I]\right)^2 \middle| \mathscr{F}_I\right]}_{\text{process variance}} + \underbrace{E\left[\left(\widehat{U}_{i,I-i} - E[\widehat{U}_{i,I}|\mathscr{F}_I]\right)^2 \middle| \mathscr{F}_I\right]}_{\text{parameter estimation error}}.\end{aligned}$$

Remark: Note that the total run-off prediction uncertainty can be clearly split (unlike the one-year prediction uncertainty) between process variance and parameter estimation error term.

In the next subsections we derive an estimator for both the above terms: process variance and parameter estimation error. We will again make use of the relationships derived in section 3.2.

5.1.1 PROCESS VARIANCE ESTIMATOR

We estimate the process variance term

$$E\left[\left(\widehat{U}_{i,I} - E[\widehat{U}_{i,I}|\mathscr{F}_I]\right)^2 \middle| \mathscr{F}_I\right]$$

by

$$\widehat{E}\left[\left(\widehat{U}_{i,I} - E[\widehat{U}_{i,I}|\mathscr{D}^I]\right)^2 \middle| \mathscr{D}^I\right] = \left[\sum_{k=I-i}^{I-1}\left(\prod_{j=I-i}^{k-1}\widehat{g}_j\right)\cdot\widehat{\sigma}_k^2\cdot\left(\prod_{l=k+1}^{I-1}\widehat{g}_l^2\right)\right]\cdot\widehat{U}_{i,I-i}, \quad i \geq 1, \qquad (5.2)$$

and we refer to appendix C.1 for the rigorous derivation of this result.

Remark:
The process variance estimator for single accident year related to our numerical example are shown in the following table

i	$\widehat{U}_{i,I-i}$	$\widehat{E}\left[\left(\widehat{U}_{i,I}-E[\widehat{U}_{i,I}\vert\mathscr{D}^I]\right)^2\middle\vert\mathscr{D}^I\right]$	$\widehat{E}\left[\left(\widehat{U}_{i,I}-E[\widehat{U}_{i,I}\vert\mathscr{D}^I]\right)^2\middle\vert\mathscr{D}^I\right]^{1/2}$
0	223'558	0	0
1	224'036	0	0
2	223'697	0	0
3	214'445	1'911	44
4	221'628	9'969	100
5	294'673	21'586	147
6	269'763	131'237	362
7	271'810	174'441	418
8	247'245	2'217'465	1'489
9	267'747	5'555'024	2'357
10	245'398	14'368'781	3'791
11	259'549	46'095'893	6'789
12	262'936	111'575'746	10'563

5.1.2 PARAMETER ESTIMATION ERROR ESTIMATOR

We estimate the parameter estimation error term

$$E\left[\left(\widehat{U}_{i,I-i}-E[\widehat{U}_{i,I}|\mathscr{F}_I]\right)^2\middle|\mathscr{F}_I\right]$$

by

$$\widehat{E}\left[\left(\widehat{U}_{i,I-i}-E[\widehat{U}_{i,I}|\mathscr{D}^I]\right)^2\middle|\mathscr{D}^I\right]=\left(1-\prod_{j=I-i}^{I-1}\widehat{g}_j\right)^2\cdot\widehat{U}_{i,I-i}^2,\quad i\geq 1, \tag{5.3}$$

and we refer to appendix C.2 for the rigorous derivation of this result.

Remark:
The parameter estimation error estimator for single accident year related to our numerical example are shown in the following table

i	$\widehat{U}_{i,I-i}$	$\widehat{E}\left[\left(\widehat{U}_{i,I-i}-E[\widehat{U}_{i,I}\vert\mathscr{D}^I]\right)^2\middle\vert\mathscr{D}^I\right]$	$\widehat{E}\left[\left(\widehat{U}_{i,I-i}-E[\widehat{U}_{i,I}\vert\mathscr{D}^I]\right)^2\middle\vert\mathscr{D}^I\right]^{1/2}$
0	223'558	0	0
1	224'036	0	0
2	223'697	0	0
3	214'445	65	8
4	221'628	1'532	39
5	294'673	6'068	78
6	269'763	16'670	129
7	271'810	33'804	184
8	247'245	53'386	231
9	267'747	121'736	349
10	245'398	69'642	264
11	259'549	1'102'668	1'050
12	262'936	36'246'911	6'021

5.1.3 TOTAL RUN-OFF PREDICTION UNCERTAINTY

Combining the above results leads to the following estimator for total run-off risk for a single accident year:

Estimator 3 (total run-off reserve risk estimator for single accident year)**.** *Under Model Assumptions 1. we have the following estimator for the total run-off prediction uncertainty for a single accident year*

$$\widehat{msep}_{U^i|\mathscr{F}_I}(\widehat{E}[U^i|\mathscr{F}_I]) = \underbrace{\left[\sum_{k=I-i}^{I-1}\left(\prod_{j=I-i}^{k-1}\widehat{g}_j\right)\cdot\widehat{\sigma}_k^2\cdot\left(\prod_{l=k+1}^{I-1}\widehat{g}_l^2\right)\right]\cdot\widehat{U}_{i,I-i}}_{\text{estimated process variance}}$$
$$+ \underbrace{\left(1-\prod_{j=I-i}^{I-1}\widehat{g}_j\right)^2\cdot\widehat{U}_{i,I-i}^2}_{\text{estimated parameter estimation error}} \quad , \quad i\geq 1. \tag{5.4}$$

Remarks:

- Note that (like for the artificial one-year reserve risk split) the estimated process variance term is a linear function in $\widehat{U}_{i,I-i}$, whereas the estimated parameter estimation error term is a quadratic function in $\widehat{U}_{i,I-i}$.
- It can be shown that the above estimator does fulfill the following "unbiasedness" property

$$\widehat{\text{msep}}_{U^i|\mathscr{F}_I}\left(\widehat{E}\left[U^i\middle|\mathscr{F}_I\right]\right) = \widehat{E}\left[\text{msep}_{U^i|\mathscr{F}_I}\left(\widehat{E}\left[U^i\middle|\mathscr{F}_I\right]\right)\middle|\mathscr{D}^I\right], \quad i\geq 1. \tag{5.5}$$

- The total run-off reserve risk estimator for single accident year related to our numerical example are shown in the following table

i	$\widehat{U}_{i,I-i}$	$\widehat{\text{msep}}_{U^i\|\mathscr{F}_I}\left(\widehat{E}\left[U^i\middle\|\mathscr{F}_I\right]\right)$	$\widehat{\text{msep}}_{U^i\|\mathscr{F}_I}\left(\widehat{E}\left[U^i\middle\|\mathscr{F}_I\right]\right)^{1/2}$
0	223'558	0	0
1	224'036	0	0
2	223'697	0	0
3	214'445	1'977	44
4	221'628	11'502	107
5	294'673	27'654	166
6	269'763	147'907	385
7	271'810	208'245	456
8	247'245	2'270'850	1'507
9	267'747	5'676'760	2'383
10	245'398	14'438'423	3'800
11	259'549	47'198'561	6'870
12	262'936	147'822'657	12'158

5.2 AGGREGATED ACCIDENT YEARS

We now take care of the aggregated accident year view. For aggregated accident years the mean square error of prediction of the total run-off prediction uncertainty is given by

$$\text{msep}_{\sum_{i=0}^I U^i|\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^I U^i\middle|\mathscr{F}_I\right]\right) := E\left[\left(\sum_{i=0}^I U^i - \widehat{E}\left[\sum_{i=0}^I U^i\middle|\mathscr{F}_I\right]\right)^2\middle|\mathscr{F}_I\right]$$

$$
\begin{aligned}
&= E\left[\left(\sum_{i=0}^{I}\left(U^i - \widehat{E}\left[U^i|\mathscr{F}_I\right]\right)\right)^2 \middle| \mathscr{F}_I\right] \\
&= \sum_{i=1}^{I} E\left[\left(U^i - \widehat{E}[U^i|\mathscr{F}_I]\right)^2 \middle| \mathscr{F}_I\right] \qquad (5.6)\\
&\quad + 2\sum_{1\le i<j\le I} E\left[\left(U^i - \widehat{E}[U^i|\mathscr{F}_I]\right)\left(U^j - \widehat{E}[U^j|\mathscr{F}_I]\right) \middle| \mathscr{F}_I\right] \\
&= \sum_{i=1}^{I} E\left[\left(U^i - \widehat{E}[U^i|\mathscr{F}_I]\right)^2 \middle| \mathscr{F}_I\right] \\
&\quad + 2\sum_{1\le i<j\le I} E\left[\left(\widehat{U}_{i,I} - \widehat{U}_{i,I-i}\right)\left(\widehat{U}_{j,I} - \widehat{U}_{j,I-j}\right) \middle| \mathscr{F}_I\right].
\end{aligned}
$$

For the above first term $\sum_{i=1}^{I} E\left[\left(U^i - \widehat{E}[U^i|\mathscr{F}_I]\right)^2 \middle| \mathscr{F}_I\right]$ we already know an appropriate estimator, since it refers to the total run-off prediction uncertainty for single accident years. Therefore it remains to estimate the second term, which we will call total run-off "covariance" term. This is done in the next subsection.

5.2.1 TOTAL RUN-OFF UNCERTAINTY ESTIMATOR (FOR AGGREGATED ACCIDENT YEARS)

We first state the following lemma:

Lemma 2. *Under Model Assumptions 1. and for $i < j$ it holds true*

$$
Cov\left(\widehat{U}_{i,I}, \widehat{U}_{j,I} \middle| \mathscr{D}^I\right) = 0. \qquad (5.7)
$$

Proof. See appendix C.3. □

Using Lemma 2. we obtain the following estimator for the total run-off "covariance" term (see appendix C.4 for more details):

$$
\begin{aligned}
&2\sum_{1\le i<j\le I} \widehat{E}\left[\left(\widehat{U}_{i,I} - \widehat{U}_{i,I-i}\right)\left(\widehat{U}_{j,I} - \widehat{U}_{j,I-j}\right) \middle| \mathscr{D}^I\right] \qquad (5.8)\\
&\qquad = 2\sum_{1\le i<j\le I}\left(1 - \prod_{k=I-i}^{I-1}\widehat{g}_k\right)\left(1 - \prod_{k=I-j}^{I-1}\widehat{g}_k\right)\cdot \widehat{U}_{i,I-i}\cdot\widehat{U}_{j,I-j}, \quad i<j.
\end{aligned}
$$

Remark:
The total run-off "covariance" term estimator related to our numerical example is given by

$$
2\sum_{1\le i<j\le I} \widehat{E}\left[\left(\widehat{U}_{i,I} - \widehat{U}_{i,I-i}\right)\left(\widehat{U}_{j,I} - \widehat{U}_{j,I-j}\right) \middle| \mathscr{D}^I\right] = 14'082'024.
$$

As a consequence we can derive estimator (1.2):

Estimator 4 (total run-off reserve risk estimator for aggregated accident years)**.** *Under Model Assumptions 1. we have the following estimator for the total run-off prediction uncertainty for aggregated accident years*

$$\widehat{msep}_{\sum_{i=0}^{I} U^i|\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^{I} U^i \middle| \mathscr{F}_I\right]\right)$$
$$= \sum_{i=1}^{I}\left\{\left[\sum_{k=I-i}^{I-1}\left(\prod_{j=I-i}^{k-1}\widehat{g}_j\right)\cdot\widehat{\sigma}_k^2\cdot\left(\prod_{l=k+1}^{I-1}\widehat{g}_l^2\right)\right]\cdot\widehat{U}_{i,I-i}+\left(1-\prod_{j=I-i}^{I-1}\widehat{g}_j\right)^2\cdot\widehat{U}_{i,I-i}^2\right\}$$
$$+2\sum_{1\le i<j\le I}\left(1-\prod_{k=I-i}^{I-1}\widehat{g}_k\right)\left(1-\prod_{k=I-j}^{I-1}\widehat{g}_k\right)\cdot\widehat{U}_{i,I-i}\cdot\widehat{U}_{j,I-j}.$$

Remarks:

- It can be shown that the above estimator does fulfill the following "unbiasedness" property

$$\widehat{\text{msep}}_{\sum_{i=0}^{I} U^i|\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^{I} U^i \middle| \mathscr{F}_I\right]\right)=\widehat{E}\left[\text{msep}_{\sum_{i=0}^{I} U^i|\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^{I} U^i \middle| \mathscr{F}_I\right]\right)\middle|\mathscr{D}^I\right]. \quad (5.9)$$

- The total run-off reserve risk estimator for aggregated accident years related to our numerical example is shown in the below table

i	$\widehat{U}_{i,I-i}$	$\widehat{\text{msep}}_{U^i\|\mathscr{F}_I}\left(\widehat{E}\left[U^i\middle\|\mathscr{F}_I\right]\right)$	$\widehat{\text{msep}}_{U^i\|\mathscr{F}_I}\left(\widehat{E}\left[U^i\middle\|\mathscr{F}_I\right]\right)^{1/2}$
0	223'558	0	0
1	224'036	0	0
2	223'697	0	0
3	214'445	1'977	44
4	221'628	11'502	107
5	294'673	27'654	166
6	269'763	147'907	385
7	271'810	208'245	456
8	247'245	2'270'850	1'507
9	267'747	5'676'760	2'383
10	245'398	14'438'423	3'800
11	259'549	47'198'561	6'870
12	262'936	147'822'657	12'158
"covariance" term		14'082'024	
	$\sum_{i=0}^{I}\widehat{U}_{i,I-i}$	$\widehat{\text{msep}}_{\sum_{i=0}^{I} U^i\|\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^{I} U^i\middle\|\mathscr{F}_I\right]\right)$	$\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I} U^i\|\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^{I} U^i\middle\|\mathscr{F}_I\right]\right)$
Total	3'226'487	231'886'560	15'228

6 COMPARISON WITH MACK'S AND MERZ-WÜTHRICH'S FORMULAS

In this section we compare our formulas with Mack's (total run-off view) and Merz-Wüthrich's (one-year view) formulas which are often used as benchmarks in the insurance industry. Since Mack's and Merz-Wüthrich's formulas are based on the standard CL methodology supported by Mack's distribution-free model, we also consider this setup throughout this section.

We recall that we denote with $C_{i,j}$ the cumulative payments for accident year $i \in \{0, \ldots, I\}$ up to development period $j \in \{0, \ldots, I\}$ and with $\mathscr{F}_k$, $k \in \{0, \ldots, 2I\}$, the total information available to the insurance company at the end of calendar year k which is given by

$$\mathscr{F}_k := \sigma\left(\{C_{i,j}\}_{i+j \le k}\right).$$

Under the standard CL framework we define the estimated ultimates $\widehat{U}_{i,j} = \widehat{E}[U^i | \mathscr{F}_{i+j}]$ as

$$\widehat{U}_{i,j} := C_{i,j} \cdot \prod_{l=j}^{I-1} \widehat{f}_l^{(i+j)}, \quad i, j \in \{0, \ldots, I\},$$

where the estimated CL factors at the end of calendar year k are given by

$$\widehat{f}_j^{(k)} := \begin{cases} \frac{\sum_{i=0}^{\min(I,k-j-1)} C_{i,j+1}}{\sum_{i=0}^{\min(I,k-j-1)} C_{i,j}}, & k - j \ge 1 \\ \bar{f}_j, & k - j < 1 \end{cases}, \quad k \in \{0, \ldots, 2I\},$$

and $\bar{f}_0, \ldots, \bar{f}_{I-1}$ are known prior expected CL factors.

6.1 APPROXIMATION RESULTS UNDER ADDITIONAL CONDITIONS

Keeping in mind the remarks done in section (3.3.1), in this section we would like to establish under which conditions Mack's and Merz-Wüthrich's formulas can be approximated by our formulas.
In other words the goal of this section is to show that, even if our Model Assumption 1 do not combine with Mack's model assumptions, under appropriate conditions our formulas deliver similar results as Mack's and Merz-Wüthrich's formulas.

First recall that, within Mack's distribution-free CL model the variance parameters are estimated by

$$\widehat{s}_j^2 := \frac{1}{I - j - 1} \sum_{i=0}^{I-j-1} C_{i,j} \left(\frac{C_{i,j+1}}{C_{i,j}} - \widehat{f}_j^{(I)} \right)^2, \quad j \in \{0, \ldots, I-2\}. \tag{6.1}$$

For $\widehat{s}_{I-1}^2$ we use the estimator proposed by Mack [3] given by

$$\widehat{s}_{I-1}^2 = \min\left\{ \widehat{s}_{I-3}^2, \widehat{s}_{I-2}^2, \frac{\widehat{s}_{I-2}^4}{\widehat{s}_{I-3}^2} \right\}. \tag{6.2}$$

We will be able to prove the following result:

Approximation Result 1. *Assume Mack's distribution-free CL model as well as*

1) no parameter estimation uncertainty,

2) $\widehat{f}_j^{(k)} \approx \bar{f}_j, \quad \forall j, k,$

3) $\widehat{\sigma}^2_{I-1} \approx \bar{f}_{I-1} \cdot \frac{\widehat{s}^2_{I-1}}{(\bar{f}_{I-1})^2}$,

where $\bar{f}_0, \ldots, \bar{f}_{I-1}$ *are known prior expected CL factors and* $\widehat{f}^{(k)}_j$ *is defined as above. Then the following approximations hold true*

$$msep^{Mack}_{\sum_{i=0}^I U^i|\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^I U^i \middle| \mathscr{F}_I\right]\right) \approx \widehat{msep}_{\sum_{i=0}^I U^i|\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^I U^i \middle| \mathscr{F}_I\right]\right), \tag{6.3}$$

$$msep^{MW}_{\sum_{i=0}^I \widehat{CDR}_i(I+1)|\mathscr{F}_I}(0) \approx \widehat{msep}_{\sum_{i=0}^I \widehat{CDR}_i(I+1)|\mathscr{F}_I}(0). \tag{6.4}$$

Proof. From the additional assumptions 1) and 2) we have

$$\widehat{g}_j \overset{1)}{=} \frac{\sum_{i=0}^{I-j-1} C_{i,j+1} \cdot \prod_{l=j+1}^{I-1} \bar{f}_l}{\sum_{i=0}^{I-j-1} C_{i,j} \cdot \prod_{l=j}^{I-1} \bar{f}_l} = \frac{1}{\bar{f}_j} \frac{\sum_{i=0}^{I-j-1} C_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}} \overset{2)}{\approx} 1, \quad j \in \{0, \ldots, I-1\}. \tag{6.5}$$

As a consequence for the one-year risk it holds true

$$\begin{aligned}
\widehat{\mathrm{msep}}_{\sum_{i=0}^I \widehat{\mathrm{CDR}}_i(I+1)|\mathscr{F}_I}(0) &\overset{(6.5)}{\approx} \sum_{i=1}^I \widehat{\sigma}^2_{I-i} \cdot \widehat{U}_{i,I-i} \overset{1)}{=} \sum_{i=1}^I \left(\widehat{U}_{i,I-i}\right)^2 \widehat{\sigma}^2_{I-i} \cdot \frac{1}{C_{i,I-i} \prod_{l=I-i}^{I-1} \bar{f}_l} \\
&\approx \sum_{i=1}^I \left(\widehat{U}_{i,I-i}\right)^2 \left(\prod_{l=I-i}^{I-1} \bar{f}_l\right) \cdot \frac{\widehat{s}^2_{I-i}}{(\bar{f}_{I-i})^2} \cdot \frac{1}{C_{i,I-i} \prod_{l=I-i}^{I-1} \bar{f}_l} \\
&= \sum_{i=1}^I \left(\widehat{U}_{i,I-i}\right)^2 \cdot \frac{\widehat{s}^2_{I-i}}{(\bar{f}_{I-i})^2} \cdot \left[\frac{1}{C_{i,I-i}}\right],
\end{aligned}$$

where in the above computation we did use the approximation

$$\begin{aligned}
\widehat{\sigma}^2_j &\overset{(6.5)}{\approx} \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} \widehat{U}_{i,j} \left(\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}} - 1\right)^2 \\
&\overset{1)}{=} \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} \widehat{U}_{i,j} \left(\frac{C_{i,j+1} \cdot \prod_{l=j+1}^{I-1} \bar{f}_l}{C_{i,j} \cdot \prod_{l=j}^{I-1} \bar{f}_l} - 1\right)^2 \\
&= \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} C_{i,j} \cdot \prod_{l=j}^{I-1} \bar{f}_l \left(\frac{C_{i,j+1}}{C_{i,j} \cdot \bar{f}_j} - 1\right)^2 \\
&= \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} C_{i,j} \cdot \prod_{l=j}^{I-1} \bar{f}_l \cdot \frac{1}{(\bar{f}_j)^2} \left(\frac{C_{i,j+1}}{C_{i,j}} - \bar{f}_j\right)^2 \\
&= \left(\frac{1}{(\bar{f}_j)^2} \cdot \prod_{l=j}^{I-1} \bar{f}_l\right) \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} C_{i,j} \left(\frac{C_{i,j+1}}{C_{i,j}} - \bar{f}_j\right)^2 \\
&\overset{2)}{\approx} \left(\prod_{l=j}^{I-1} \bar{f}_l\right) \cdot \frac{\widehat{s}^2_j}{(\bar{f}_j)^2}, \quad j \in \{0, \ldots, I-2\},
\end{aligned}$$

and the additional assumption 3) i.e. $\widehat{\sigma}^2_{I-1} \approx \bar{f}_{I-1} \cdot \frac{\widehat{s}^2_{I-1}}{(\bar{f}_{I-1})^2}$.

Therefore the first approximation is proved since Merz-Wüthrich's formula under no parameter estimation uncertainty (see Merz-Wüthrich [6]) reduces to

$$\mathrm{msep}^{\mathrm{MW}}_{\sum_{i=0}^I \widehat{\mathrm{CDR}}_i(I+1)|\mathscr{F}_I}(0) = \sum_{i=1}^I \left(\widehat{U}_{i,I-i}\right)^2 \frac{\widehat{s}^2_{I-i}}{\left(\widehat{f}^{(I)}_{I-i}\right)^2} \left[\frac{1}{C_{i,I-i}}\right].$$

For the total run-off risk it holds true

$$\begin{aligned}\widehat{\mathrm{msep}}_{\sum_{i=0}^{I} U^i|\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^{I} U^i \middle| \mathscr{F}_I\right]\right) &\overset{(6.5)}{\approx} \sum_{i=1}^{I}\left[\sum_{j=I-i}^{I-1}\widehat{\sigma}_j^2\right]\cdot\widehat{U}_{i,I-i} \\ &\overset{1)}{=} \sum_{i=1}^{I}\left(\widehat{U}_{i,I-i}\right)^2\cdot\sum_{j=I-i}^{I-1}\widehat{\sigma}_j^2\cdot\left[\frac{1}{C_{i,I-i}\prod_{l=I-i}^{I-1}\bar{f}_l}\right] \\ &\approx \sum_{i=1}^{I}\left(\widehat{U}_{i,I-i}\right)^2\cdot\sum_{j=I-i}^{I-1}\left(\prod_{l=j}^{I-1}\bar{f}_l\right)\cdot\frac{\widehat{s}_j^2}{\left(\bar{f}_j\right)^2}\cdot\left[\frac{1}{C_{i,I-i}\prod_{l=I-i}^{I-1}\bar{f}_l}\right],\end{aligned}$$

where in the last step of the above computation we did use again the approximation

$$\widehat{\sigma}_j^2 \approx \left(\prod_{l=j}^{I-1}\bar{f}_l\right)\cdot\frac{\widehat{s}_j^2}{\left(\bar{f}_j\right)^2}, \quad j\in\{0,\ldots,I-1\}.$$

Therefore the second approximation is proved since Mack's formula under no parameter estimation uncertainty reduces to

$$\mathrm{msep}^{\mathrm{Mack}}_{\sum_{i=0}^{I} U^i|\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^{I} U^i \middle| \mathscr{F}_I\right]\right) = \sum_{i=1}^{I}\left(\widehat{U}_{i,I-i}\right)^2\sum_{j=I-i}^{I-1}\frac{\widehat{s}_j^2}{\left(\widehat{f}_j^{(I)}\right)^2}\left[\frac{1}{C_{i,I-i}\cdot\prod_{l=I-i}^{j-1}\widehat{f}_l^{(I)}}\right].$$

□

Remarks:

- The assumptions we have taken in order to be able to prove the above stated approximations are very strong.
- In the above proof, due to the fact that Mack's distribution-free CL model does fulfil the unconditional unbiased property, we should have even better done the comparison using the parameter estimators given by

$$\widehat{g}_j = 1, \quad j\in\{0,\ldots,I-1\},$$
$$\widehat{\sigma}_j^2 = \frac{1}{I-j}\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}\left(\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}}-1\right)^2, \quad j\in\{0,\ldots,I-2\}.$$

  In that respect, note that the stated approximations would anyway be well fulfilled.

In the next section, for didactic purposes based on a toy numerical example, we will relax assumptions 1) and 3), i.e. only assume the stability conditions $\widehat{f}_j^{(k)} \approx \bar{f}_j$, $\forall j,k$, to hold true, and compare again the result obtained by applying our formulas with the result obtained by applying Mack's and Merz-Wüthrich's formulas.

6.2 TOY NUMERICAL EXAMPLE

We will consider the following claims payments data ($C_{i,j}$):

i/j	0	1	2	3	4
0	2'357	7'432	12'444	16'639	16'738
1	8'345	26'046	43'651	56'832	
2	5'492	16'799	26'999		
3	7'688	23'695			
4	4'566				

The related estimated CL factors $(\widehat{f}_j^{(4)})$ can be computed to be given by

j	0	1	2	3
$\widehat{f}_j^{(4)}$	3.097	1.653	1.310	1.006

and the CL variance parameter estimators

$$\widehat{s}_j^2 := \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} C_{i,j} \left(\frac{C_{i,j+1}}{C_{i,j}} - \widehat{f}_j^{(4)} \right)^2, \quad j \in \{0, \ldots, I-2\}$$

$$\widehat{s}_{I-1}^2 := \min \left\{ \widehat{s}_{I-3}^2, \widehat{s}_{I-2}^2, \frac{\widehat{s}_{I-2}^4}{\widehat{s}_{I-3}^2} \right\}$$

can be computed to be given by

j	0	1	2	3
$\widehat{s}_j^2$	7.340	26.173	11.962	5.467

In the next subsection we will consider three choices of prior expected CL factors and compute for each choice the prediction uncertainties.

6.2.1 CHOICE 1

We choose the prior expected factors $(\bar{f}_j)$ equal to the estimated CL factors $(\widehat{f}_j^{(4)})$. Then we obtain the following series of estimated CL factors

j	0	1	2	3
$\bar{f}_j$	3.097	1.653	1.310	1.006
$\widehat{f}_j^{(0)}$	3.097	1.653	1.310	1.006
$\widehat{f}_j^{(1)}$	3.153	1.653	1.310	1.006
$\widehat{f}_j^{(2)}$	3.128	1.674	1.310	1.006
$\widehat{f}_j^{(3)}$	3.105	1.676	1.337	1.006
$\widehat{f}_j^{(4)}$	3.097	1.653	1.310	1.006

and the triangle of estimated ultimates $(\widehat{U}_{i,j})$ is given by

i/j	0	1	2	3	4
0	15'897	16'184	16'396	16'738	16'738
1	57'298	57'460	58'713	57'170	
2	37'901	37'861	35'573		
3	53'794	51'597			
4	30'796				

The estimated parameters $(\widehat{g}_j)$ and $(\widehat{\sigma}_j^2)$ in our model are therefore given by

j	0	1	2	3
$\widehat{g}_j$	0.9891	0.9926	0.9840	1.0000
$\widehat{\sigma}_j^2$	25.3279	81.1887	28.5140	10.0143

where for $\widehat{\sigma}_3^2$ we use the estimator proposed by Mack [3] given by

$$\widehat{\sigma}_3^2 = \min\left\{\widehat{\sigma}_1^2, \widehat{\sigma}_2^2, \frac{\widehat{\sigma}_2^4}{\widehat{\sigma}_1^2}\right\}.$$

Finally, the estimated prediction uncertainties can then be computed to be:

	One-year risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^I \widehat{\text{CDR}}_i(I+1)\mid\mathscr{F}_I}(0)$	Total run-off risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^I U^i\mid\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^I U^i \middle\mid \mathscr{F}_I\right]\right)$
our Model	2'864	4'490
underlying CL Model	3'629	4'114

This choice of the prior expected factors is rather artificial. In the next subsection we will do a more realistic choice.

6.2.2 CHOICE 2

We choose the prior expected factors $(\bar{f}_j)$ to be in line with the estimated CL factors $(\widehat{f}_j^{(4)})$. Then we obtain the following series of estimated CL factors

j	0	1	2	3
$\bar{f}_j$	3.100	1.700	1.300	1.010
$\widehat{f}_j^{(0)}$	3.100	1.700	1.300	1.010
$\widehat{f}_j^{(1)}$	3.153	1.700	1.300	1.010
$\widehat{f}_j^{(2)}$	3.128	1.674	1.300	1.010
$\widehat{f}_j^{(3)}$	3.105	1.676	1.337	1.010
$\widehat{f}_j^{(4)}$	3.097	1.653	1.310	1.006

and the triangle of estimated ultimates $(\widehat{U}_{i,j})$ is given by

i/j	0	1	2	3	4
0	16'309	16'589	16'339	16'805	16'738
1	58'734	57'261	58'950	57'170	
2	37'770	38'013	35'573		
3	54'011	51'597			
4	30'796				

The estimated parameters $(\widehat{g}_j)$ and $(\widehat{\sigma}_j^2)$ in our model are therefore given by

j	0	1	2	3
$\widehat{g}_j$	0.9798	0.9910	0.9826	0.9960
$\widehat{\sigma}_j^2$	27.7894	100.6560	44.1363	19.3532

Finally, the estimated prediction uncertainties can then be computed to be:

	One-year risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^I \widehat{\text{CDR}}_i(I+1)\mid\mathscr{F}_I}(0)$	Total run-off risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^I U^i\mid\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^I U^i \middle\mid \mathscr{F}_I\right]\right)$
our Model	3'530	5'808
underlying CL Model	3'629	4'114

6.2.3 CHOICE 3

We choose the prior expected factors $(\bar{f}_j)$ to be less similar to the estimated CL factors $(\widehat{f}_j^{(4)})$ than in choice 2. Then we obtain the following series of estimated CL factors

j	0	1	2	3
$\bar{f}_j$	3.000	1.750	1.250	1.000
$\widehat{f}_j^{(0)}$	3.000	1.750	1.250	1.000
$\widehat{f}_j^{(1)}$	3.153	1.750	1.250	1.000
$\widehat{f}_j^{(2)}$	3.128	1.674	1.250	1.000
$\widehat{f}_j^{(3)}$	3.105	1.676	1.337	1.000
$\widehat{f}_j^{(4)}$	3.097	1.653	1.310	1.006

and the triangle of estimated ultimates $(\widehat{U}_{i,j})$ is given by

i/j	0	1	2	3	4
0	15'468	16'258	15'555	16'639	16'738
1	57'560	54'514	58'366	57'170	
2	35'957	37'637	35'573		
3	53'476	51'597			
4	30'796				

The estimated parameters $(\widehat{g}_j)$ and $(\widehat{\sigma}_j^2)$ in our model are therefore given by

j	0	1	2	3
$\widehat{g}_j$	0.9849	1.0100	0.9985	1.0059
$\widehat{\sigma}_j^2$	102.9613	202.4903	99.8821	49.2687

Finally, the estimated prediction uncertainties can then be computed to be:

	One-year risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^I \widehat{\text{CDR}}_i(I+1)\mid\mathscr{F}_I}(0)$	Total run-off risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^I U^i\mid\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^I U^i \middle\vert \mathscr{F}_I\right]\right)$
our Model	4'484	6'487
underlying CL Model	3'629	4'114

Concluding remark:
Looking at the above results we observe that if the prior expected CL factors do not differ too much from the estimated CL factors then the magnitude order of the estimated prediction uncertainties under the two different model assumptions is similar.

Moreover, since within Mack's model the unconditional unbiased property is fulfilled, for comparison purposes we should better evaluate our formulas using the modified parameter estimators as explained in section 3.1. When doing this, we note that the results are even more aligned:

	One-year risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^I \widehat{\text{CDR}}_i(I+1)\mid\mathscr{F}_I}(0)$	Total run-off risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^I U^i\mid\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^I U^i \middle\vert \mathscr{F}_I\right]\right)$
our Model	4'484	6'487
our Model with $(\widehat{g}_j = 1)$ and adjusted $(\widehat{\sigma}_j^2)$	3'554	4'811
underlying CL Model	3'629	4'114

Remark:
Coming back to our main numerical example we have to note, that the ultimate estimates triangle considered has been derived starting from the following claims payments data $(C_{i,j})$

i/j	0	1	2	3	4	5	6	7	8	9	10	11	12
0	157'495	212'770	219'680	220'978	221'276	223'724	223'743	223'765	223'669	223'601	223'558	223'558	223'558
1	154'969	213'352	219'201	220'469	222'751	223'958	224'005	224'030	223'975	224'048	224'036	224'036	
2	152'833	209'969	214'692	220'040	223'467	223'754	223'752	223'593	223'585	223'688	223'697		
3	144'223	207'644	212'443	214'108	214'661	214'610	214'564	214'484	214'459	214'459			
4	145'612	209'604	214'161	215'982	217'962	220'783	221'078	221'614	221'616				
5	196'695	282'621	288'676	290'036	292'206	294'531	294'671	294'705					
6	181'381	260'308	266'497	269'130	269'404	269'691	269'720						
7	177'168	263'130	268'848	270'787	271'624	271'688							
8	156'505	230'607	237'102	244'847	245'940								
9	157'839	239'723	261'213	264'755									
10	159'429	233'309	239'800										
11	169'990	246'019											
12	173'377												

by applying, constantly over time, the standard CL methodology, i.e. using the following series of estimated CL factors

j	0	1	2	3	4	5	6	7	8	9	10	11
$\bar{f}_j$	1.4000	1.0400	1.0100	1.0050	1.0050	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\widehat{f}_j^{(0)}$	1.4000	1.0400	1.0100	1.0050	1.0050	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\widehat{f}_j^{(1)}$	1.4000	1.0400	1.0100	1.0050	1.0050	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\widehat{f}_j^{(2)}$	1.4000	1.0325	1.0100	1.0050	1.0050	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\widehat{f}_j^{(3)}$	1.4000	1.0299	1.0059	1.0050	1.0050	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\widehat{f}_j^{(4)}$	1.4000	1.0275	1.0058	1.0014	1.0050	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\widehat{f}_j^{(5)}$	1.3949	1.0264	1.0121	1.0058	1.0111	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\widehat{f}_j^{(6)}$	1.4036	1.0255	1.0111	1.0091	1.0082	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\widehat{f}_j^{(7)}$	1.4086	1.0246	1.0106	1.0075	1.0059	1.0001	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000
$\widehat{f}_j^{(8)}$	1.4190	1.0245	1.0093	1.0078	1.0044	1.0001	1.0001	0.9996	1.0000	1.0000	1.0000	1.0000
$\widehat{f}_j^{(9)}$	1.4248	1.0241	1.0094	1.0078	1.0061	1.0000	0.9998	0.9997	0.9997	1.0000	1.0000	1.0000
$\widehat{f}_j^{(10)}$	1.4339	1.0245	1.0091	1.0067	1.0065	1.0003	0.9998	0.9998	1.0000	0.9998	1.0000	1.0000
$\widehat{f}_j^{(11)}$	1.4366	1.0312	1.0117	1.0062	1.0056	1.0003	1.0003	0.9998	1.0002	0.9999	1.0000	1.0000
$\widehat{f}_j^{(12)}$	1.4375	1.0309	1.0119	1.0060	1.0049	1.0003	1.0003	0.9998	1.0001	0.9999	1.0000	1.0000

Therefore, based on the claims payments triangle we can evaluate both Mack's and Merz-Wüthrich's formulas and compare them with our results:

	One-year risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}\widehat{\text{CDR}}_i(I+1)\vert\mathscr{F}_I}(0)$	Total run-off risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}U^i\vert\mathscr{F}_I}\left(\widehat{E}\left[\sum_{i=0}^{I}U^i\middle\vert\mathscr{F}_I\right]\right)$
our Model	12'025	15'228
our Model with $(\widehat{g}_j = 1)$ and adjusted $(\widehat{\sigma}_j^2)$	11'080	13'687
underlying CL Model	11'203 (Mack)	13'457 (Merz-Wüthrich)

7 CONCLUSION

In actuarial practice a wide number of reserving methodologies are applied. Unfortunately not all these reserving methodologies are supported by a fully well defined stochastic reserving model and for some methodologies that are supported by such a stochastic model the reserve risk uncertainties estimators may be unknown.

In all of these cases it is currently not possible to properly estimate the reserve risk uncertainties.

In our paper we did provide a solution to overcome this problem since we derived estimators that only depend on the historical triangle of ultimate estimates and can therefore be evaluated even if the reserving methodology is, from a pure stochastic point of view, not fully well defined. We therefore believe that our formulas could be particularly useful in order to estimate the reserve risk uncertainties (both one-year and total run-off) of a given insurance portfolio in case

(a) the prediction uncertainties related to the stochastic model supporting the applied reserving methodology are not yet known

(b) we have a probabilistic lack of consistency in the stochastic model supporting the applied reserving methodology which could be due for instance (as often usual in practice) to

  - different basics reserving methodologies applied to different accident years (for example Bornhuetter-Ferguson methodology for less mature accident years and Chain-Ladder methodology for more mature accident years),
  - mixtures of basics methodologies applied to specific accident years,

(c) we do not exactly know according to which reserving methodology the ultimate estimates have been generated (for example if the reserving analysis is done on a more granular level than the reserve risk uncertainties need to be quantified for solvency purposes).

Finally, we also believe that, if for a given insurance portfolio a well defined reserving methodology is applied (for which a fully consistent and validated supporting model exists and the prediction uncertainties estimators within that model are known and fulfill good properties), then the corresponding estimators (without forgetting to allow also for a model risk loading) should be preferred to those presented in this paper.
Nevertheless, based on numerical examples, we did show that in the standard CL framework and under stability conditions, our "high-level" formulas deliver similar results as Mack's and Merz-Wüthrich's formulas.

APPENDIX

A.1 UNBIASEDNESS OF THE ESTIMATORS $\widehat{g}_j$ and $\widehat{\sigma}_j^2$

In this section we prove in detail the conditional unbiasedness of the estimators $\widehat{g}_j$ and $\widehat{\sigma}_j^2$.

It holds true

$$\begin{aligned}
E\left[\widehat{g}_j\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right] &= E\left[\frac{\sum_{i=0}^{I-j-1}\widehat{U}_{i,j+1}}{\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}}\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right]\\
&= \frac{1}{\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}}E\left[\sum_{i=0}^{I-j-1}\widehat{U}_{i,j+1}\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right]\\
&= \frac{1}{\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}}\sum_{i=0}^{I-j-1}E\left[E\left[\widehat{U}_{i,j+1}\middle|\mathscr{D}_j^{j+i}\right]\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right]\\
&= \frac{1}{\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}}\sum_{i=0}^{I-j-1}E\left[g_j\cdot\widehat{U}_{i,j}\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right] = g_j, \quad j\in\{0,\dots,I-1\},
\end{aligned}$$

and

$$\begin{aligned}
&E\left[\widehat{\sigma}_j^2\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right] = E\left[\frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}\left(\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}}-\widehat{g}_j\right)^2\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right]\\
&= \frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}E\left[\left(\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}}-g_j\right)^2+(\widehat{g}_j-g_j)^2-2\left(\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}}-g_j\right)(\widehat{g}_j-g_j)\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right]\\
&= \frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}Var\left(\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}}\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right)\\
&\quad+\frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}E\left[(\widehat{g}_j-g_j)^2\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right]\\
&\quad-\frac{1}{I-j-1}E\left[2\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}\left(\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}}-g_j\right)(\widehat{g}_j-g_j)\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right]\\
&= \frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}\cdot\frac{\sigma_j^2}{\widehat{U}_{i,j}}\\
&\quad+\frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}E\left[(\widehat{g}_j-g_j)^2\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right]\\
&\quad-\frac{1}{I-j-1}\left(\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}\right)\cdot E\left[2\left(\frac{\sum_{i=0}^{I-j-1}\widehat{U}_{i,j+1}}{\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}}-g_j\right)(\widehat{g}_j-g_j)\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right]\\
&= \frac{1}{I-j-1}(I-j)\sigma_j^2-\frac{1}{I-j-1}\left(\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}\right)\cdot E\left[(\widehat{g}_j-g_j)^2\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right]\\
&= \frac{1}{I-j-1}(I-j)\sigma_j^2-\frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}\cdot Var\left(\widehat{g}_j\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right)\\
&= \frac{1}{I-j-1}(I-j)\sigma_j^2-\frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}\cdot Var\left(\sum_{i=0}^{I-j-1}\frac{\widehat{U}_{i,j}}{\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}}\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}}\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right)\\
&= \frac{1}{I-j-1}(I-j)\sigma_j^2-\frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}\cdot\sum_{i=0}^{I-j-1}\frac{\widehat{U}_{i,j}^2}{\left(\sum_{i=0}^{I-j-1}\widehat{U}_{i,j}\right)^2}Var\left(\frac{\widehat{U}_{i,j+1}}{\widehat{U}_{i,j}}\middle|\sigma\left(\bigcup_{i=0}^{I}\{\widehat{U}_{i,0},\dots,\widehat{U}_{i,j}\}\right)\right)\\
&= \frac{1}{I-j-1}(I-j)\sigma_j^2-\frac{1}{I-j-1}\sigma_j^2=\sigma_j^2, \quad j\in\{0,\dots,I-2\}.
\end{aligned}$$

B.1 ONE-YEAR UNCERTAINTY ESTIMATOR FOR SINGLE ACCIDENT YEAR

We estimate the one-year uncertainty for single accident year $E\left[\left(\widehat{U}_{i,I-i+1}-\widehat{U}_{i,I-i}\right)^2\middle|\mathscr{F}_I\right]$ by

$$\begin{aligned}
&\widehat{E}\left[\left(\widehat{U}_{i,I-i+1}-\widehat{U}_{i,I-i}\right)^2\middle|\mathscr{D}^I\right]\\
&=\widehat{E}\left[\widehat{U}^2_{i,I-i+1}\middle|\mathscr{D}^I\right]-2\widehat{E}\left[\widehat{U}_{i,I-i+1}\widehat{U}_{i,I-i}\middle|\mathscr{D}^I\right]+\widehat{E}\left[\widehat{U}^2_{i,I-i}\middle|\mathscr{D}^I\right]\\
&=\left(\widehat{Var}\left(\widehat{U}_{i,I-i+1}\middle|\mathscr{D}^I\right)+\widehat{E}\left[\widehat{U}_{i,I-i+1}\middle|\mathscr{D}^I\right]^2\right)\\
&-2\left[\widehat{E}\left[\widehat{U}_{i,I-i+1}\middle|\mathscr{D}^I\right]\widehat{E}\left[\widehat{U}_{i,I-i}\middle|\mathscr{D}^I\right]+\widehat{Cov}\left(\widehat{U}_{i,I-i+1},\widehat{U}_{i,I-i}\middle|\mathscr{D}^I\right)\right]\\
&\quad+\left(\underbrace{\widehat{Var}\left(\widehat{U}_{i,I-i}\middle|\mathscr{D}^I\right)}_{=0}+\widehat{E}\left[\widehat{U}_{i,I-i}\middle|\mathscr{D}^I\right]^2\right)\\
&=\left(\widehat{\sigma}^2_{I-i}\cdot\widehat{U}_{i,I-i}+\widehat{U}^2_{i,I-i}\cdot\widehat{g}^2_{I-i}\right)-2\cdot\left[\widehat{U}^2_{i,I-i}\cdot\widehat{g}_{I-i}+\underbrace{\widehat{Cov}\left(\widehat{U}_{i,I-i+1},\widehat{U}_{i,I-i}\middle|\mathscr{D}^I\right)}_{=0}\right]+\widehat{U}^2_{i,I-i}\\
&=\widehat{\sigma}^2_{I-i}\cdot\widehat{U}_{i,I-i}+\left(\widehat{g}_{I-i}-1\right)^2\cdot\widehat{U}^2_{i,I-i},\quad i\geq 1.
\end{aligned}$$

B.2 SPLIT OF THE ONE-YEAR PREDICTION UNCERTAINTY

For a single accident year $i \geq 1$ the mean square error of prediction of the one-year prediction uncertainty can be split into three components (process variance, parameter estimation uncertainty and mixed term) as follows:

$$\begin{aligned}
&\mathrm{msep}_{\widehat{\mathrm{CDR}}_i(I+1)|\mathscr{F}_I}(0):=E\left[\left(\widehat{E}[U^i|\mathscr{F}_{I+1}]-\widehat{E}[U^i|\mathscr{F}_I]-0\right)^2\middle|\mathscr{F}_I\right]\\
&=E\left[\left(\widehat{E}[U^i|\mathscr{F}_{I+1}]-E[U^i|\mathscr{F}_{I+1}]+E[U^i|\mathscr{F}_I]-\widehat{E}[U^i|\mathscr{F}_I]+E[U^i|\mathscr{F}_{I+1}]-E[U^i|\mathscr{F}_I]\right)^2\middle|\mathscr{F}_I\right]\\
&=E\left[\left(E[U^i|\mathscr{F}_{I+1}]-E\left[E[U^i|\mathscr{F}_{I+1}]\middle|\mathscr{F}_I\right]\right)^2\middle|\mathscr{F}_I\right]+E\left[\left(\widehat{U}_{i,I-i+1}-E[U^i|\mathscr{F}_{I+1}]\right)^2\middle|\mathscr{F}_I\right]\\
&\quad+E\left[\left(E[U^i|\mathscr{F}_I]-\widehat{U}_{i,I-i}\right)^2\middle|\mathscr{F}_I\right]\\
&\quad+2\left(E[U^i|\mathscr{F}_I]-\widehat{U}_{i,I-i}\right)E\left[\left(\widehat{U}_{i,I-i+1}-E[U^i|\mathscr{F}_{I+1}]\right)\middle|\mathscr{F}_I\right]\\
&\quad+2\left(E[U^i|\mathscr{F}_I]-\widehat{U}_{i,I-i}\right)\underbrace{E\left[\left(E[U^i|\mathscr{F}_{I+1}]-E[U^i|\mathscr{F}_I]\right)\middle|\mathscr{F}_I\right]}_{=0}\\
&\quad+2E\left[\left(\widehat{U}_{i,I-i+1}-E[U^i|\mathscr{F}_{I+1}]\right)\left(E[U^i|\mathscr{F}_{I+1}]-E[U^i|\mathscr{F}_I]\right)\middle|\mathscr{F}_I\right]\\
&=\underbrace{Var\left(E[U^i|\mathscr{F}_{I+1}]\middle|\mathscr{F}_I\right)}_{\text{process variance}}\\
&\quad+\underbrace{E\left[\left(\widehat{U}_{i,I-i+1}-E[U^i|\mathscr{F}_{I+1}]\right)^2\middle|\mathscr{F}_I\right]+E\left[\left(\widehat{U}_{i,I-i}-E[U^i|\mathscr{F}_I]\right)^2\middle|\mathscr{F}_I\right]}_{\text{parameter estimation uncertainty: first term}}
\end{aligned}$$

$$- \underbrace{2E\left[\left(\widehat{U}_{i,I-i} - E[U^i|\mathscr{F}_I]\right)\left(\widehat{U}_{i,I-i+1} - E[U^i|\mathscr{F}_{I+1}]\right)\middle|\mathscr{F}_I\right]}_{\text{parameter estimation uncertainty: second term}}$$

$$+ \underbrace{2E\left[\left(\widehat{U}_{i,I-i+1} - E[U^i|\mathscr{F}_{I+1}]\right)\left(E[U^i|\mathscr{F}_{I+1}] - E[U^i|\mathscr{F}_I]\right)\middle|\mathscr{F}_I\right]}_{\text{mixed term}}.$$

B.3 PROCESS VARIANCE ESTIMATOR

We estimate the process variance $Var\left(E[U^i|\mathscr{F}_{I+1}]\middle|\mathscr{F}_I\right)$ by

$$\begin{aligned}
\widehat{Var}\left(E[U^i|\mathscr{D}^{I+1}]\middle|\mathscr{D}^I\right) &= \widehat{Var}\left(E[\widehat{U}_{i,I}|\mathscr{D}^{I+1}]\middle|\mathscr{D}^I\right) \\
&= \widehat{Var}\left(\widehat{U}_{i,I-i+1}\prod_{j=I-i+1}^{I-1} g_j\middle|\mathscr{D}^I\right) \\
&= \left(\prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)^2 \cdot \widehat{Var}\left(\widehat{U}_{i,I-i+1}\middle|\mathscr{D}^I\right) \\
&= \left(\prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)^2 \cdot \widehat{\sigma}^2_{I-i}\cdot\widehat{U}_{i,I-i}, \quad i \geq 1.
\end{aligned}$$

B.4 PARAMETER ESTIMATION UNCERTAINTY ESTIMATOR

We estimate the first term of the parameter estimation uncertainty

$$E\left[\left(\widehat{U}_{i,I-i+1} - E[U^i|\mathscr{F}_{I+1}]\right)^2\middle|\mathscr{F}_I\right] + E\left[\left(\widehat{U}_{i,I-i} - E[U^i|\mathscr{F}_I]\right)^2\middle|\mathscr{F}_I\right]$$

by

$$\begin{aligned}
&\widehat{E}\left[\left(\widehat{U}_{i,I-i+1} - E[U^i|\mathscr{D}^{I+1}]\right)^2\middle|\mathscr{D}^I\right] + \widehat{E}\left[\left(\widehat{U}_{i,I-i} - E[U^i|\mathscr{D}^I]\right)^2\middle|\mathscr{D}^I\right] \\
&= \widehat{E}\left[\left(\widehat{U}_{i,I-i+1} - \widehat{U}_{i,I-i+1}\prod_{j=I-i+1}^{I-1} g_j\right)^2\middle|\mathscr{D}^I\right] + \widehat{E}\left[\left(\widehat{U}_{i,I-i}\prod_{j=I-i}^{I-1} g_j - \widehat{U}_{i,I-i}\right)^2\middle|\mathscr{D}^I\right] \\
&= \left(1 - \prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)^2 \cdot \widehat{E}\left[\widehat{U}^2_{i,I-i+1}\middle|\mathscr{D}^I\right] + \left(\prod_{j=I-i}^{I-1}\widehat{g}_j - 1\right)^2 \cdot \widehat{U}^2_{i,I-i} \\
&= \left(1 - \prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)^2 \cdot \left(\widehat{Var}\left(\widehat{U}_{i,I-i+1}\middle|\mathscr{D}^I\right) + \widehat{E}\left[\widehat{U}_{i,I-i+1}\middle|\mathscr{D}^I\right]^2\right) + \left(\prod_{j=I-i}^{I-1}\widehat{g}_j - 1\right)^2 \cdot \widehat{U}^2_{i,I-i} \\
&= \left(1 - \prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)^2 \cdot \left(\widehat{\sigma}^2_{I-i}\cdot\widehat{U}_{i,I-i} + \widehat{g}^2_{I-i}\cdot\widehat{U}^2_{i,I-i}\right) + \left(\prod_{j=I-i}^{I-1}\widehat{g}_j - 1\right)^2 \cdot \widehat{U}^2_{i,I-i} \\
&= \left(1 - \prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)^2 \cdot \widehat{\sigma}^2_{I-i}\cdot\widehat{U}_{i,I-i} + \left(\widehat{g}_{I-i} - \prod_{j=I-i}^{I-1}\widehat{g}_j\right)^2 \cdot \widehat{U}^2_{i,I-i} + \left(1 - \prod_{j=I-i}^{I-1}\widehat{g}_j\right)^2 \cdot \widehat{U}^2_{i,I-i}, \quad i \geq 1.
\end{aligned}$$

We estimate the second term of the parameter estimation uncertainty

$$-2E\left[\left(\widehat{U}_{i,I-i} - E[U^i|\mathscr{F}_I]\right)\left(\widehat{U}_{i,I-i+1} - E[U^i|\mathscr{F}_{I+1}]\right)\middle|\mathscr{F}_I\right]$$

by

$$
\begin{aligned}
&-2\widehat{E}\left[\left(\widehat{U}_{i,I-i}-E[U^i|\mathscr{D}^I]\right)\left(\widehat{U}_{i,I-i+1}-E[U^i|\mathscr{D}^{I+1}]\right)\middle|\mathscr{D}^I\right]\\
&\quad=-2\widehat{E}\left[\left(\widehat{U}_{i,I-i}-\widehat{U}_{i,I-i}\prod_{j=I-i}^{I-1}g_j\right)\left(\widehat{U}_{i,I-i+1}-\widehat{U}_{i,I-i+1}\prod_{j=I-i+1}^{I-1}g_j\right)\middle|\mathscr{D}^I\right]\\
&\quad=-2\left(1-\prod_{j=I-i}^{I-1}\widehat{g}_j\right)\left(1-\prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)\cdot\widehat{U}_{i,I-i}\cdot\widehat{E}\left[\widehat{U}_{i,I-i+1}\middle|\mathscr{D}^I\right]\\
&\quad=-2\left(1-\prod_{j=I-i}^{I-1}\widehat{g}_j\right)\left(\widehat{g}_{I-i}-\prod_{j=I-i}^{I-1}\widehat{g}_j\right)\cdot\widehat{U}^2_{i,I-i},\quad i\geq 1.
\end{aligned}
$$

B.5 MIXED TERM ESTIMATOR

We estimate the mixed term $2E\left[\left(\widehat{U}_{i,I-i+1}-E[U^i|\mathscr{F}_{I+1}]\right)\left(E[U^i|\mathscr{F}_{I+1}]-E[U^i|\mathscr{F}_I]\right)\middle|\mathscr{F}_I\right]$ by

$$
\begin{aligned}
&2\widehat{E}\left[\left(\widehat{U}_{i,I-i+1}-E[U^i|\mathscr{D}^{I+1}]\right)\left(E[U^i|\mathscr{D}^{I+1}]-E[U^i|\mathscr{D}^I]\right)\middle|\mathscr{D}^I\right]\\
&\quad=2\widehat{E}\left[\left(\widehat{U}_{i,I-i+1}-\widehat{U}_{i,I-i+1}\prod_{j=I-i+1}^{I-1}g_j\right)\left(\widehat{U}_{i,I-i+1}\prod_{j=I-i+1}^{I-1}g_j-\widehat{U}_{i,I-i}\prod_{j=I-i}^{I-1}g_j\right)\middle|\mathscr{D}^I\right]\\
&\quad=2\left(1-\prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)\cdot\widehat{E}\left[\widehat{U}_{i,I-i+1}\left(\widehat{U}_{i,I-i+1}\prod_{j=I-i+1}^{I-1}g_j-\widehat{U}_{i,I-i}\prod_{j=I-i}^{I-1}g_j\right)\middle|\mathscr{D}^I\right]\\
&\quad=2\left(1-\prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)\left(\prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)\cdot\widehat{E}\left[\widehat{U}^2_{i,I-i+1}\middle|\mathscr{D}^I\right]\\
&\qquad-2\left(1-\prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)\cdot\widehat{U}_{i,I-i}\cdot\left(\prod_{j=I-i}^{I-1}\widehat{g}_j\right)\cdot\widehat{E}\left[\widehat{U}_{i,I-i+1}\middle|\mathscr{D}^I\right]\\
&\quad=2\left(1-\prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)\left(\prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)\cdot\left(\widehat{\sigma}^2_{I-i}\cdot\widehat{U}_{i,I-i}+\widehat{g}^2_{I-i}\cdot\widehat{U}^2_{i,I-i}\right)\\
&\qquad-2\left(1-\prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)\cdot\left(\prod_{j=I-i}^{I-1}\widehat{g}_j\right)\cdot\widehat{g}_{I-i}\cdot\widehat{U}^2_{i,I-i}\\
&\quad=2\left(1-\prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)\left(\prod_{j=I-i+1}^{I-1}\widehat{g}_j\right)\cdot\widehat{\sigma}^2_{I-i}\cdot\widehat{U}_{i,I-i},\quad i\geq 1
\end{aligned}
$$

B.6 PROOF OF LEMMA 1.

We remind that $\widehat{U}_{i,I-i}$ is $\mathscr{D}^I$-measurable and for $i<j$ we also have that $\widehat{U}_{j,I-j+1}$ is $\mathscr{D}^I_{I-i}$-measurable. Therefore for $i<j$ it holds true:

$$
\begin{aligned}
&Cov\left(\widehat{U}_{i,I-i+1},\widehat{U}_{j,I-j+1}\middle|\mathscr{D}^I\right)\\
&\quad=E\left[Cov\left(\widehat{U}_{i,I-i+1},\underbrace{\widehat{U}_{j,I-j+1}}_{\mathscr{D}^I_{I-i}\text{-measurable}}\middle|\mathscr{D}^I_{I-i}\right)\middle|\mathscr{D}^I\right]
\end{aligned}
$$

$$+ Cov\left(E[\widehat{U}_{i,I-i+1}|\mathscr{D}^I_{I-i}], E[\widehat{U}_{j,I-j+1}|\mathscr{D}^I_{I-i}]\middle|\mathscr{D}^I\right)$$

$$= E\left[0\middle|\mathscr{D}^I\right] + Cov\left(g_{I-i}\cdot \underbrace{\widehat{U}_{i,I-i}}_{\mathscr{D}^I\text{-measurable}}, \widehat{U}_{j,I-j+1}\middle|\mathscr{D}^I\right) = 0+0=0.$$

B.7 ONE-YEAR UNCERTAINTY ESTIMATOR FOR AGGREGATED ACCIDENT YEAR

Using Lemma 1. we estimate the term

$$E\left[\left(\widehat{E}[U^i|\mathscr{F}_{I+1}] - \widehat{E}[U^i|\mathscr{F}_I]\right)\left(\widehat{E}[U^j|\mathscr{F}_{I+1}] - \widehat{E}[U^j|\mathscr{F}_I]\right)\middle|\mathscr{F}_I\right], \quad i<j,$$

by

$$\begin{aligned}
&\widehat{E}\left[\left(\widehat{U}_{i,I-i+1} - \widehat{U}_{i,I-i}\right)\left(\widehat{U}_{j,I-j+1} - \widehat{U}_{j,I-j}\right)\middle|\mathscr{D}^I\right]\\
=&\widehat{E}\left[\widehat{U}_{i,I-i+1}\widehat{U}_{j,I-j+1}\middle|\mathscr{D}^I\right] - \widehat{E}\left[\widehat{U}_{i,I-i+1}\middle|\mathscr{D}^I\right]\widehat{U}_{j,I-j} - \widehat{U}_{i,I-i}\widehat{E}\left[\widehat{U}_{j,I-j+1}\middle|\mathscr{D}^I\right] + \widehat{U}_{i,I-i}\widehat{U}_{j,I-j}\\
=&\widehat{E}\left[\widehat{U}_{i,I-i+1}\middle|\mathscr{D}^I\right]\cdot\widehat{E}\left[\widehat{U}_{j,I-j+1}\middle|\mathscr{D}_I\right] - \widehat{E}\left[\widehat{U}_{i,I-i+1}\middle|\mathscr{D}^I\right]\widehat{U}_{j,I-j}\\
&\quad - \widehat{U}_{i,I-i}\widehat{E}\left[\widehat{U}_{j,I-j+1}\middle|\mathscr{D}^I\right] + \widehat{U}_{i,I-i}\widehat{U}_{j,I-j}\\
=&\widehat{g}_{I-i}\cdot\widehat{U}_{i,I-i}\cdot\widehat{g}_{I-j}\cdot\widehat{U}_{j,I-j} - \widehat{g}_{I-i}\cdot\widehat{U}_{i,I-i}\widehat{U}_{j,I-j} - \widehat{U}_{i,I-i}\cdot\widehat{g}_{I-j}\cdot\widehat{U}_{j,I-j} + \widehat{U}_{i,I-i}\widehat{U}_{j,I-j}\\
=&\widehat{g}_{I-i}\cdot\widehat{g}_{I-j}\cdot\widehat{U}_{i,I-i}\widehat{U}_{j,I-j} - \widehat{g}_{I-i}\cdot\widehat{U}_{i,I-i}\widehat{U}_{j,I-j} - \widehat{g}_{I-j}\cdot\widehat{U}_{i,I-i}\widehat{U}_{j,I-j} + \widehat{U}_{i,I-i}\widehat{U}_{j,I-j}\\
=&\left(\widehat{g}_{I-i}\cdot\widehat{g}_{I-j} - \widehat{g}_{I-i} - \widehat{g}_{I-j} + 1\right)\cdot\widehat{U}_{i,I-i}\cdot\widehat{U}_{j,I-j}, \quad i<j.
\end{aligned}$$

C.1 PROCESS VARIANCE ESTIMATOR FOR SINGLE ACCIDENT YEAR

It holds true

$$\begin{aligned}
&\widehat{E}\left[\left(\widehat{U}_{i,I}-E[\widehat{U}_{i,I}|\mathscr{D}^I]\right)^2\Big|\mathscr{D}^I\right]\\
&=\widehat{Var}\left(\widehat{U}_{i,I}\Big|\mathscr{D}^I\right)\\
&=\widehat{E}\left[Var\left(\widehat{U}_{i,I}|\mathscr{D}_{I-1}^{I-1+i}\right)|\mathscr{D}^I\right]+\widehat{Var}\left(E\left[\widehat{U}_{i,I}|\mathscr{D}_{I-1}^{I-1+i}\right]|\mathscr{D}^I\right)\\
&=\widehat{E}\left[\sigma_{I-1}^2\cdot\widehat{U}_{i,I-1}|\mathscr{D}^I\right]+\widehat{Var}\left(g_{I-1}\cdot\widehat{U}_{i,I-1}|\mathscr{D}^I\right)\\
&=\widehat{\sigma}_{I-1}^2\cdot\widehat{U}_{i,I-i}\cdot\prod_{j=I-i}^{I-2}\widehat{g}_j+\widehat{g}_{I-1}^2\cdot\widehat{Var}(\widehat{U}_{i,I-1}|\mathscr{D}^I)\\
&=\widehat{\sigma}_{I-1}^2\cdot\widehat{U}_{i,I-i}\cdot\prod_{j=I-i}^{I-2}\widehat{g}_j+\widehat{g}_{I-1}^2\cdot\left(\widehat{\sigma}_{I-2}^2\cdot\widehat{U}_{i,I-i}\cdot\prod_{j=I-i}^{I-3}\widehat{g}_j+\widehat{g}_{I-2}^2\cdot\widehat{Var}(\widehat{U}_{i,I-2}|\mathscr{D}^I)\right)\\
&=\widehat{\sigma}_{I-1}^2\cdot\widehat{U}_{i,I-i}\cdot\prod_{j=I-i}^{I-2}\widehat{g}_j+\widehat{g}_{I-1}^2\cdot\widehat{\sigma}_{I-2}^2\cdot\widehat{U}_{i,I-i}\cdot\prod_{j=I-i}^{I-3}\widehat{g}_j+\widehat{g}_{I-1}^2\cdot\widehat{g}_{I-2}^2\cdot\widehat{Var}(\widehat{U}_{i,I-2}|\mathscr{D}^I)\\
&=\cdots=\left[\sum_{k=I-i}^{I-1}\left(\prod_{j=I-i}^{k-1}\widehat{g}_j\right)\cdot\widehat{\sigma}_k^2\cdot\left(\prod_{l=k+1}^{I-1}\widehat{g}_l^2\right)\right]\cdot\widehat{U}_{i,I-i},\quad i\geq 1,
\end{aligned}$$

where the last step follows by iteration of the same procedure applied in the prior steps until we reach the triangle diagonal.

C.2 PARAMETER ESTIMATION ERROR ESTIMATOR FOR SINGLE ACCIDENT YEAR

It holds true

$$\begin{aligned}
\widehat{E}\left[\left(\widehat{U}_{i,I-i}-E[\widehat{U}_{i,I}|\mathscr{D}^I]\right)^2\Big|\mathscr{D}^I\right]&=\widehat{E}\left[\left(\widehat{U}_{i,I-i}-\widehat{U}_{i,I-i}\prod_{j=I-i}^{I-1}g_j\right)^2\Big|\mathscr{D}^I\right]\\
&=\widehat{E}\left[\widehat{U}_{i,I-i}^2\cdot\left(1-\prod_{j=I-i}^{I-1}g_j\right)^2\Big|\mathscr{D}^I\right]\\
&=\left(1-\prod_{j=I-i}^{I-1}\widehat{g}_j\right)^2\cdot\widehat{U}_{i,I-i}^2,\quad i\geq 1.
\end{aligned}$$

C.3 PROOF OF LEMMA 2.

For $i<j$ it holds that $\widehat{U}_{i,I}$ is $\mathscr{D}_{I-1}^{I-1+j}$-measurable and therefore we have:

$$\begin{aligned}
&Cov\left(\widehat{U}_{i,I},\widehat{U}_{j,I}\Big|\mathscr{D}^I\right)\\
&=E\left[Cov\left(\underbrace{\widehat{U}_{i,I}}_{\mathscr{D}_{I-1}^{I-1+j}\text{-measurable}},\widehat{U}_{j,I}\Big|\mathscr{D}_{I-1}^{I-1+j}\right)\Big|\mathscr{D}^I\right]\\
&\quad+Cov\left(E[\widehat{U}_{i,I}|\mathscr{D}_{I-1}^{I-1+j}],E[\widehat{U}_{j,I}|\mathscr{D}_{I-1}^{I-1+j}]\Big|\mathscr{D}^I\right)
\end{aligned}$$

$$= E\left[0\middle|\mathscr{D}^I\right] + Cov\left(\widehat{U}_{i,I}, g_{I-1}\cdot\widehat{U}_{j,I-1}\middle|\mathscr{D}^I\right)$$
$$= g_{I-1}\cdot Cov\left(\widehat{U}_{i,I}, \widehat{U}_{j,I-1}\middle|\mathscr{D}^I\right) = \cdots = 0,$$

where the last step follows by iteration of the same procedure applied in the prior steps (until we reach the triangle diagonal).

C.4 TOTAL RUN-OFF UNCERTAINTY ESTIMATOR FOR AGGREGATED ACCIDENT YEAR

Using Lemma 2. we estimate the term

$$E\left[\left(\widehat{U}_{i,I}-\widehat{U}_{i,I-i}\right)\left(\widehat{U}_{j,I}-\widehat{U}_{j,I-j}\right)\middle|\mathscr{F}_I\right], \quad i<j,$$

by

$$\widehat{E}\left[\left(\widehat{U}_{i,I}-\widehat{U}_{i,I-i}\right)\left(\widehat{U}_{j,I}-\widehat{U}_{j,I-j}\right)\middle|\mathscr{D}^I\right]$$
$$= \widehat{E}\left[\widehat{U}_{i,I}\widehat{U}_{j,I}\middle|\mathscr{D}^I\right] - \widehat{E}\left[\widehat{U}_{i,I}\widehat{U}_{j,I-j}\middle|\mathscr{D}^I\right] - \widehat{E}\left[\widehat{U}_{i,I-i}\widehat{U}_{j,I}\middle|\mathscr{D}^I\right] + \widehat{E}\left[\widehat{U}_{i,I-i}\widehat{U}_{j,I-j}\middle|\mathscr{D}^I\right]$$
$$= \widehat{E}\left[\widehat{U}_{i,I}\middle|\mathscr{D}^I\right]\widehat{E}\left[\widehat{U}_{j,I}\middle|\mathscr{D}^I\right] - \widehat{U}_{j,I-j}\cdot\widehat{E}\left[\widehat{U}_{i,I}\middle|\mathscr{D}^I\right] - \widehat{U}_{i,I-i}\cdot\widehat{E}\left[\widehat{U}_{j,I}\middle|\mathscr{D}^I\right] + \widehat{U}_{i,I-i}\widehat{U}_{j,I-j}$$
$$= \widehat{E}\left[\widehat{U}_{i,I}\middle|\mathscr{D}^I\right]\left(\widehat{E}\left[\widehat{U}_{j,I}\middle|\mathscr{D}^I\right] - \widehat{U}_{j,I-j}\right) - \widehat{U}_{i,I-i}\cdot\left(\widehat{E}\left[\widehat{U}_{j,I}\middle|\mathscr{D}^I\right] - \widehat{U}_{j,I-j}\right)$$
$$= \left(\widehat{U}_{i,I-i}\cdot\prod_{k=I-i}^{I-1}\widehat{g}_k - \widehat{U}_{i,I-i}\right)\left(\widehat{U}_{j,I-j}\cdot\prod_{k=I-j}^{I-1}\widehat{g}_k - \widehat{U}_{j,I-j}\right)$$
$$= \left(1-\prod_{k=I-i}^{I-1}\widehat{g}_k\right)\left(1-\prod_{k=I-j}^{I-1}\widehat{g}_k\right)\cdot\widehat{U}_{i,I-i}\cdot\widehat{U}_{j,I-j}, \quad i<j.$$

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