Prediction Error of the Future Claims Component of Premium Liabilities under the Loss Ratio Approach

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ABSTRACT

In this paper we construct a stochastic model and derive approximation formulae to estimate the standard error of prediction under the loss ratio approach of assessing premium liabilities. We focus on the future claims component of premium liabilities and examine the weighted and simple average loss ratio estimators. The resulting mean square error of prediction contains the process error component and the estimation error component, in which the former refers to future claims variability while the latter refers to the uncertainty in parameter estimation. We illustrate the application of our model to public liability data and simulated data.

KEYWORDS

Premium liabilities, loss ratio, standard error of prediction, mean square error of prediction

1. Introduction

There has been extensive literature on loss reserving models over the past 25 years, including the Mack (1993) model. While the focus has been largely on how to tackle outstanding claims liabilities, relatively few materials have been presented for premium liabilities. Some references include Cantin and Trahan (1999), Buchanan (2002), Collins and Hu (2003), and Yan (2005), which focus on the central estimate (i.e., the mean) of premium liabilities but not on the underlying variability.

As noted in Clark et al. (2003), the International Accounting Standards Board (IASB) has proposed a new reporting regime for insurance contracts, in which both outstanding claims liabilities and premium liabilities should be assessed at their fair values. It is generally understood that this fair value includes a "margin" allowing for different types of variability for insurance liabilities. Accordingly, the Australian Prudential Regulation Authority (APRA) has prescribed an approach similar to the fair value approach. Under Prudential Standard GPS 310, a "risk margin" has to be explicitly calculated such that outstanding claims liabilities and premium liabilities are assessed at a sufficiency level of 75%, subject to a minimum of the mean plus one-half the standard deviation. Australian Accounting Standard AASB 1023 also requires inclusion of a risk margin, though there is no prescription on the adequacy level. No matter what approach one takes, it is obvious that urgency for developing proper tools to measure liability variability exists not only for outstanding claims liabilities but also for premium liabilities. In addition, according to Yan (2005), premium liabilities account for around 30% of insurance liabilities for direct insurers and 15% to 20% for reinsurers in Australia from 2002 to 2004. Premium liabilities represent a significant portion of an insurer's liabilities and proper assessment of the underlying variability should not be overlooked.

The definition of premium liabilities varies for different countries. Broadly speaking, premium liabilities refer to all future claim payments and associated expenses arising from future events after the valuation date which are insured under the existing unexpired policies. Buchanan (2002) notes that there are two main methods of determining the central estimate of premium liabilities. The first method is prospective in nature and involves a full actuarial assessment from first principles. Yan (2005) calls this method the claims approach and differentiates it into the loss ratio approach and historical claims approach. The loss ratio approach is the most common one for premium liability assessment in practice and is essentially an extension of the outstanding claims liability valuation. It applies a projected loss ratio to the unearned premiums or number of policies unexpired. The historical claims approach uses the number of claims and average claim size and is more suitable for short-tailed lines of business where data is sufficient. While the historical claims approach has been studied extensively under the classical risk theory, the loss ratio approach has received relatively little attention in the literature. In this paper we follow the loss ratio approach and attempt to supplement this knowledge gap.

On the other hand, the second method noted in Buchanan (2002) is retrospective in nature and involves an adjustment of the unearned premiums to take out the profit margin. As discussed in Cantin and Trahan (1999) and Yan (2005), both Canadian and Australian accounting standards require a reporting of this unearned premiums item, in which a premium deficiency reserve is added if this item is less than the premium liability estimate determined by the first method. Obviously the first method above plays a key role in premium liability assessment, and we focus on the loss ratio approach under this prospective method.

In this paper we construct a stochastic model to estimate the standard error of prediction under the loss ratio approach of assessing premium liabilities. We focus on modeling the future claims which form the largest component in premium liabilities (about 85% according to Collins and Hu 2003). We look at the weighted average ultimate loss ratio and simple average ultimate loss ratio, and derive approximation formulae to estimate the corresponding mean square error of prediction with respect to the accident year following the valuation date. As similarly reasoned in Taylor (2000), the resulting mean square error of prediction is composed of the process error component and the estimation error component, and no covariance term exists as one part is related only to the future while the other only to the past. We also illustrate the application of our model to Australian private-sector direct insurers' public liability data and some hypothetical data simulated from the compound Poisson model.

The outline of the paper is as follows. In Section 2 we introduce the basic notation and assumptions of our model. In Section 3 we present the formulae for estimating the standard error of prediction for premium liabilities. In Section 4 we apply the model to public liability data and simulated data and analyze the results. In Section 5 we set forth our concluding remarks. Appendices A to D furnish the proofs for the formulae stated in this paper.

2. Notation and assumptions

Let $C_{i,j}$ (for $1 \leq i \leq n+1$ and $1 \leq j \leq n$) be a random variable representing the cumulative claim amount (either paid or incurred) of accident year i and development year j. Assuming all claims are settled in n years, $C_{i,n}$ represents the ultimate claim amount of accident year i. We consider the case where a run-off triangle of $C_{i,j}$'s is available for $i + j \leq n+1$. In effect, the valuation date is at the end of accident year n, $C_{i,j}$'s for $i + j > n+1$ and $1 \leq i \leq n$ refer to the future claims of outstanding claims liabilities, and $C_{n+1,j}$'s refer to the future claims of premium liabilities. Let E_i (for $1 \leq i \leq n+1$) be the premiums of accident year i. The premiums are assumed to be known. The term $C_{i,n}/E_i$ then becomes the ultimate loss ratio of accident year i.

It is also assumed that exposure is evenly distributed over each year, and the exposure distribution of accident year $n+1$ is the same as that of the past accident years. In reality, the future exposure relating to premium liabilities would arise more from the earlier part of accident year $n+1$, while the past exposure would spread more uniformly across the whole year. Although the timing of claims development is actually different between the two cases, the way that the claims develop to ultimate remains basically the same. As our focus is on the ultimate loss ratio, this approximation is reasonable and represents a convenient simplification for the model setting.

As mentioned in the Introduction, the loss ratio approach for the premium liability valuation is basically an extension of the outstanding claims liability valuation. Hence we start with the structure of the chain ladder method, which is the most common method for assessing outstanding claims liabilities in practice and is linked to a distribution-free model in Mack (1993). Incorporating E_i into the three basic assumptions of the Mack (1993) model, we deduce the following for $1 \leq i \leq n+1$:

$$\mathrm{E}\left(\frac{C_{i,j+1}}{E_i}\,\middle|\, C_{i,1}, C_{i,2}, \ldots, C_{i,j}\right) = \frac{C_{i,j}}{E_i} f_j; \quad (\text{for} \quad 1 \leq j \leq n-1) \qquad (2.1)$$

$$\mathrm{Var}\left(\frac{C_{i,j+1}}{E_i}\,\middle|\, C_{i,1}, C_{i,2}, \ldots, C_{i,j}\right) = \frac{C_{i,j}}{E_i^2} \sigma_j^2; \quad (\text{for} \quad 1 \leq j \leq n-1) \qquad (2.2)$$

$$C_{i,j} \text{ and } C_{g,h} \text{ are independent.} \quad (\text{for } i \neq g) \qquad (2.3)$$

The parameter f_j is the development ratio and the parameter σ_j^2 is related to the conditional variance of $C_{i,j+1}$. These two parameters are unknown and need to be estimated from the claims data.

As of the valuation date, there is no claims data for accident year $n+1$. In order to model the future claims in the first development year of accident year $n+1$, we add the following two assumptions for $1 \le i \le n+1$, which are analogous to those for new claims in Schnieper (1991):

$$\mathrm{E}\left(\frac{C_{i,1}}{E_i}\right) = u; \qquad (2.4)$$

$$\mathrm{Var}\left(\frac{C_{i,1}}{E_i}\right) = \frac{v^2}{E_i}. \qquad (2.5)$$

Rearranging assumptions (2.4) and (2.5) into $\mathrm{E}(C_{i,1}) = E_i u$ and $\mathrm{Var}(C_{i,1}) = E_i v^2$, we can see that the mean and variance of the claim amount of the first development year is effectively assumed to be proportional to the premiums. This is analogous to assumptions (2.1) and (2.2), in which the conditional mean and variance of the claim amount $C_{i,j}$ for $2 \le j \le n$ depends on the previous development year's claim amount $C_{i,j-1}$. The parameters u and v^2 are unknown and can be estimated from the claims and premiums data.

Mack (1993) suggests the following unbiased estimators for f_j and σ_j^2 and proves that $\hat{f}_j$ and $\hat{f}_h$ are uncorrelated for $j \ne h$:

$$\hat{f}_j = \frac{\sum_{r=1}^{n-j} C_{r,j+1}}{\sum_{r=1}^{n-j} C_{r,j}} = \frac{\sum_{r=1}^{n-j} C_{r,j}\dfrac{C_{r,j+1}}{C_{r,j}}}{\sum_{r=1}^{n-j} C_{r,j}}; \quad (\text{for} \quad 1 \le j \le n-1) \qquad (2.6)$$

$$\hat{\sigma}_j^2 = \frac{1}{n-j-1}\sum_{r=1}^{n-j} C_{r,j}\left(\frac{C_{r,j+1}}{C_{r,j}} - \hat{f}_j\right)^2; \quad (\text{for} \quad 1 \le j \le n-2)$$

$$\hat{\sigma}_{n-1}^2 = \min\left(\frac{\hat{\sigma}_{n-2}^4}{\hat{\sigma}_{n-3}^2}, \hat{\sigma}_{n-3}^2\right). \qquad (2.7)$$

We now introduce two unbiased estimators for u and v^2 as follows, which are again based on Schnieper (1991):

$$\hat{u} = \frac{\sum_{r=1}^{n} C_{r,1}}{\sum_{r=1}^{n} E_r} = \frac{\sum_{r=1}^{n} E_r \dfrac{C_{r,1}}{E_r}}{\sum_{r=1}^{n} E_r}; \qquad (2.8)$$

$$\hat{v}^2 = \frac{1}{n-1}\sum_{r=1}^{n} E_r\left(\frac{C_{r,1}}{E_r} - \hat{u}\right)^2. \qquad (2.9)$$

It can be seen that both formulae (2.6) and (2.8) are weighted averages and that both formulae (2.7) and (2.9) use weighted sums of squares. The proofs for unbiasedness of $\hat{u}$ and $\hat{v}^2$ are given in Appendix A.

In effect, we integrate the model assumptions in Mack (1993) with those of development year one for new claims in Schnieper (1991). The overall structure is based on the chain ladder method. It then becomes possible to assess the next accident year's ultimate loss ratio using the observed run-off triangle. As shown in the next section, the results of the outstanding claims liability valuation (i.e., projected ultimate loss ratios of the past accident years) are carried through to the premium liability valuation (regarding the expected ultimate loss ratio of the accident year following the valuation date).

3. Standard error of prediction

In practice, actuaries often examine the projected ultimate loss ratios of past accident years and compare these figures with target or budget ratios or industry ratios to obtain an estimate of the next accident year's ultimate loss ratio. Here we assume no such prior knowledge or objective information is available and investigate the following two estimators for the next accident year's expected ultimate loss ratio $q = \mathrm{E}(C_{n+1,n}/E_{n+1})$:

$$\hat{q} = \frac{\sum_{i=1}^{n} C_{i,n+1-i}\hat{f}_{n+1-i}\hat{f}_{n+2-i}\cdots\hat{f}_{n-1}}{\sum_{i=1}^{n} E_i} = \frac{\sum_{i=1}^{n} C_{i,n+1-i}S_{n+1-i,n-1}}{\sum_{i=1}^{n} E_i} = \frac{\sum_{i=1}^{n} \hat{C}_{i,n}}{\sum_{i=1}^{n} E_i} = \frac{\sum_{i=1}^{n} E_i \dfrac{\hat{C}_{i,n}}{E_i}}{\sum_{i=1}^{n} E_i}; \qquad (3.1)$$

$$\hat{q}^* = \frac{1}{n}\sum_{i=1}^{n}\frac{C_{i,n+1-i}\hat{f}_{n+1-i}\hat{f}_{n+2-i}\cdots\hat{f}_{n-1}}{E_i} = \frac{1}{n}\sum_{i=1}^{n}\frac{C_{i,n+1-i}S_{n+1-i,n-1}}{E_i} = \frac{1}{n}\sum_{i=1}^{n}\frac{\hat{C}_{i,n}}{E_i}. \quad (3.2)$$

Let $S_{j,h} = \hat{f}_j\hat{f}_{j+1}\ldots\hat{f}_h$ (for $j \le h$; equal to one otherwise) and $\hat{C}_{i,j} = C_{i,n+1-i}S_{n+1-i,j-1}$ (for $i + j > n+1$ and $1 \le i \le n$). $\hat{C}_{1,n}$ is read as equal to $C_{1,n}$. Formula (3.1) gives a weighted average while formula (3.2) provides a simple average. Both estimators are unbiased and the proofs are set forth in Appendix B. For the expected future claims component of premium liabilities of the next accident year $\mathrm{E}(C_{n+1,n})$, we define its estimator as

$$\hat{C}_{n+1,n} = E_{n+1}\hat{q}. \quad (3.3)$$

For now we deal with (3.1) and, as shown later, the results of (3.1) can readily be extended to the use of (3.2). We will also look at the effects of excluding some accident years when calculating q, as a practitioner may exclude or adjust a few years' projected loss ratios that are regarded as inconsistent with the rest, out of date, or irrelevant. Such circumstances arise when there have been past changes in, for example, the regulations, underwriting procedures, claims management, business mix, or reinsurance arrangements.

Using the idea in Chapter 6 of Taylor (2000), we define the mean square error of prediction of the estimator $\hat{q}$ as follows:

$$\mathrm{MSEP}(\hat{q}) = \mathrm{E}\left(\left(\frac{C_{n+1,n}}{E_{n+1}} - \hat{q}\right)^2\right) = \mathrm{E}\left(\left(\frac{C_{n+1,n}}{E_{n+1}} - q + q - \hat{q}\right)^2\right) = \mathrm{E}\left(\left(\frac{C_{n+1,n}}{E_{n+1}} - q\right)^2\right) + \mathrm{E}((\hat{q}-q)^2)$$

($C_{n+1,n}$ and $\hat{q}$ are independent due to (2.3))

$$= \mathrm{Var}\left(\frac{C_{n+1,n}}{E_{n+1}}\right) + \mathrm{Var}(\hat{q}). \qquad (\hat{q} \text{ is unbiased}) \quad (3.4)$$

The mean square error of prediction above consists of two components: the first allows for process error and the second for estimation error. The process error component refers to future claims variability and the estimation error component refers to the uncertainty in parameter estimation due to sampling error. As similarly noted in Taylor (2000), there is no covariance term in (3.4) because at the valuation date, $C_{n+1,n}$ is entirely related to the future while $\hat{q}$ is completely based on the past observations.

The corresponding standard error of prediction can then be calculated as

$$\mathrm{SEP}(\hat{q}) = \sqrt{\mathrm{MSEP}(\hat{q})}. \quad (3.5)$$

For the next accident year's expected ultimate claim amount, we compute the standard error of prediction of its estimator as

$$\mathrm{SEP}(\hat{C}_{n+1,n}) = E_{n+1}\ \mathrm{SEP}(\hat{q}) = E_{n+1}\sqrt{\mathrm{MSEP}(\hat{q})}. \quad (3.6)$$

We derive the process error component as follows and the proof is given in Appendix C:

$$\mathrm{Var}\left(\frac{C_{n+1,n}}{E_{n+1}}\right) = \frac{1}{E_{n+1}}\mathrm{E}\left(\frac{C_{n+1,n}}{E_{n+1}}\right) \times \sum_{j=1}^{n-1}\frac{\sigma_j^2}{f_j}f_{j+1}f_{j+2}\cdots f_{n-1} + \frac{\nu^2}{E_{n+1}}f_1^2f_2^2\ldots f_{n-1}^2, \quad (3.7)$$

which can be estimated by

$$\widehat{\mathrm{Var}}\left(\frac{C_{n+1,n}}{E_{n+1}}\right) = \frac{\hat{q}}{E_{n+1}}\sum_{j=1}^{n-1}\frac{\hat{\sigma}_j^2}{\hat{f}_j}S_{j+1,n-1} + \frac{\hat{\nu}^2}{E_{n+1}}S_{1,n-1}^2. \quad (3.8)$$

The estimation error component requires more computation. We derive the following approximation for this component and the proof is provided in Appendix D:

$$\begin{aligned}\mathrm{Var}(\hat{q}) \approx & \frac{1}{\left(\sum_{i=1}^{n} E_i\right)^2} \sum_{j=1}^{n-1} \left(\sum_{i=n+1-j}^{n} \frac{\mathrm{E}(C_{i,n})}{f_j} \right)^2 \mathrm{Var}(\hat{f}_j) \\ & + \frac{1}{\left(\sum_{i=1}^{n} E_i\right)^2} \sum_{i=1}^{n} f_{n+1-i}^2 f_{n+2-i}^2 \cdots f_{n-1}^2 \mathrm{Var}(C_{i,n+1-i}) \\ & + \frac{2}{\left(\sum_{i=1}^{n} E_i\right)^2} \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} \left(\sum_{r=n+1-j}^{n} \frac{\mathrm{E}(C_{r,n})}{f_j} \right) \\ & \times (f_{n+1-i} f_{n+2-i} \cdots f_{n-1}) \mathrm{Cov}(\hat{f}_j, C_{i,n+1-i}), \end{aligned} \quad (3.9)$$

which can be estimated by

$$\begin{aligned}\widehat{\mathrm{Var}}(\hat{q}) = & \frac{1}{\left(\sum_{i=1}^{n} E_i\right)^2} \sum_{j=1}^{n-1} \left(\sum_{i=n+1-j}^{n} \frac{\hat{C}_{i,n}}{\hat{f}_j} \right)^2 \widehat{\mathrm{Var}}(\hat{f}_j) \\ & + \frac{1}{\left(\sum_{i=1}^{n} E_i\right)^2} \sum_{i=1}^{n} S_{n+1-i,n-1}^2 \widehat{\mathrm{Var}}(C_{i,n+1-i}) \\ & + \frac{2}{\left(\sum_{i=1}^{n} E_i\right)^2} \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} \sum_{r=n+1-j}^{n} \frac{\hat{C}_{r,n}}{\hat{f}_j} \\ & \times S_{n+1-i,n-1} \widehat{\mathrm{Cov}}(\hat{f}_j, C_{i,n+1-i}), \end{aligned} \quad (3.10)$$

where the estimators of the variance and covariance terms are derived as

$$\widehat{\mathrm{Var}}(\hat{f}_j) = \frac{\hat{\sigma}_j^2}{\sum_{r=1}^{n-j} C_{r,j}}; \quad (3.11)$$

$$\widehat{\mathrm{Var}}(C_{i,n+1-i}) = C_{i,n+1-i} \sum_{j=1}^{n-i} \frac{\hat{\sigma}_j^2}{\hat{f}_j} S_{j+1,n-i} + E_i \hat{v}^2 S_{1,n-i}^2; \quad (3.12)$$

$$\widehat{\mathrm{Cov}}(\hat{f}_j, C_{i,n+1-i}) = \frac{C_{i,n+1-i}}{\sum_{r=1}^{n-j} C_{r,j}} \frac{\hat{\sigma}_j^2}{\hat{f}_j}. \quad (3.13)$$

By now we have shown all the formulae that are needed to calculate the standard error of prediction of (3.1). Note that the term $f_j f_{j+1} \cdots f_{n-1}$ for $j > n-1$ is read as equal to one in the summations. In the next section we will apply these formulae to some real claims data and simulated data.

4. Illustrative examples

We first apply the formulae shown previously to some public liability data. We use the aggregated claim payments and premiums (both gross and net of reinsurance) of the private-sector direct insurers from the "Selected Statistics on the General Insurance Industry" (APRA) for accident years 1981 to 1991 ($n = 10$). Adopting the approach as described in Hart et al. (1996), all the figures have been converted to constant dollar values in accordance with the average weekly ordinary time earnings (AWOTE) before the calculations. This procedure is common in practice and is based on the assumption that wage inflation is the 'normal' inflation for the claims.

The inflation-adjusted claims (incremental) and premiums data are presented in Table 1 below for gross of reinsurance and in Table 2 for net of reinsurance. All the figures are in thousands.

The two run-off triangles show that it takes several years for public liability claims to develop and this line of business is generally regarded as a long-tailed line of business. We use formulae (2.6) to (2.9), (3.1), and (3.3) to estimate the parameters, accident year 1991's expected ultimate loss ratio, and so the expected future claims of premium liabilities. We then adopt formulae (3.4) to (3.13) to compute the corresponding standard error of prediction. Table 3 below presents our results both gross and net of reinsurance.

As shown in Table 3, the estimated gross and net expected ultimate loss ratios for accident year 1991 are 49.2% and 53.6%. The standard error of prediction for the future claims of premium liabilities, expressed as a percentage of the mean, is greater for gross than for net. The gross and net percentages are 47.1% and 33.1% respectively. This feature is in line with the general perception that gross liability variability is greater than its net counterpart. In both cases the process error component is larger than the estimation error component.

Table 1. Public liability (gross of reinsurance)

Claims	1	2	3	4	5	6	7	8	9	10	Premiums
1981	15,898	20,406	17,189	19,627	35,034	12,418	8,922	12,555	8,965	6,693	289,732
1982	16,207	21,518	17,753	18,780	19,113	18,634	15,857	13,050	9,362		319,216
1983	14,141	20,315	16,458	25,473	16,427	92,888	18,698	15,295			314,607
1984	14,649	21,162	19,084	23,857	20,171	15,098	17,637				344,446
1985	21,949	26,455	23,285	25,251	22,286	23,424					418,358
1986	18,989	28,741	32,754	30,240	28,443						535,658
1987	19,367	36,420	31,204	27,487							639,130
1988	26,860	39,550	33,852								751,897
1989	23,738	52,683									780,669
1990	34,567										719,181
1991											334,566

Table 2. Public liability (net of reinsurance)

Claims	1	2	3	4	5	6	7	8	9	10	Premiums
1981	13,451	16,801	12,947	13,752	13,802	8,583	6,847	9,237	5,641	3,784	168,975
1982	13,533	17,489	13,111	13,541	13,603	11,937	10,524	8,609	5,987		186,990
1983	11,808	17,525	12,644	15,609	11,821	17,305	10,524	11,061			200,475
1984	13,309	17,806	14,777	17,295	15,340	12,060	11,752				222,843
1985	19,546	22,786	19,686	21,860	19,268	18,692					262,748
1986	17,865	25,888	28,194	25,578	22,985						333,716
1987	17,797	33,517	24,182	24,337							410,429
1988	24,591	33,398	28,512								502,869
1989	21,567	46,146									532,298
1990	30,343										545,218
1991											234,659

All accident years' estimated ultimate loss ratios are fairly consistent with one another except the gross loss ratio of accident year 1983. A closer look at the claims data reveals that the gross claim payments made at accident year 1983 and development year 6 are \$92,888 thousand, the amount of which is much larger than the other figures in the same development year. We find that if the amount is changed to say \$18,000 thousand, then the standard error of prediction reduces significantly from 47.1% to 35.5%. Whether to allow for this extra variability or adjust the data is a matter of judgment and in practice requires further investigation into the underlying features of those claims.

As mentioned in the previous section, one can exclude some accident years' loss ratios when calculating q if those loss ratios are considered inconsistent, out of date, or irrelevant. This computation can readily be done by setting an indicator variable for each accident year, in which the indicator is one if the loss ratio of that accident year is included and zero otherwise. Table 4 below demonstrates some results of using different numbers of accident years in computing q and SEP($\hat{q}$) with (3.1), (3.8), and (3.10).

For each case of a particular number of accident years being included, Table 4 sets out the average figures across all the possible combinations of accident years in that case. It can be seen that the estimation error component and so the standard error of prediction decreases when more accident years (i.e., more data) are used. The process error component is stable because in our analysis, the indicator adjustments are only applied to (3.1) and (3.10) but not (2.6) to (2.9).

Hitherto we have been focusing on the use of (3.1). In many situations one may prefer using a simple average of loss ratios as in (3.2). We only need to replace (3.9) and (3.10) with the following, the proof of which is analogous to

Table 3. Estimated results using weighted average loss ratio

Gross of Reinsurance				Net of Reinsurance			
Accident Year	Premiums	Ultimate Claims	Loss Ratio	Accident Year	Premiums	Ultimate Claims	Loss Ratio
1981	289,732	157,705	54.4%	1981	168,975	104,844	62.0%
1982	319,216	156,934	49.2%	1982	186,990	112,391	60.1%
1983	314,607	244,292	77.6%	1983	200,475	118,959	59.3%
1984	344,446	159,365	46.3%	1984	222,843	124,138	55.7%
1985	418,358	192,494	46.0%	1985	262,748	165,031	62.8%
1986	535,658	247,328	46.2%	1986	333,716	191,706	57.4%
1987	639,130	259,865	40.7%	1987	410,429	195,706	47.7%
1988	751,897	313,187	41.7%	1988	502,869	227,747	45.3%
1989	780,669	364,832	46.7%	1989	532,298	264,892	49.8%
1990	719,181	421,727	58.6%	1990	545,218	297,641	54.6%
1991	334,566	164,750	49.2%	1991	234,659	125,678	53.6%
$\text{Var}\left(\frac{C_{n+1,n}}{E_{n+1}}\right)$	$\text{Var}(\hat{q})$	$\text{SEP}(\hat{q})$	$\frac{\text{SEP}(\hat{C}_{n+1,n})}{\text{E}(C_{n+1,n})}$	$\text{Var}\left(\frac{C_{n+1,n}}{E_{n+1}}\right)$	$\text{Var}(\hat{q})$	$\text{SEP}(\hat{q})$	$\frac{\text{SEP}(\hat{C}_{n+1,n})}{\text{E}(C_{n+1,n})}$
0.0481	0.0058	0.2322	47.1%	0.0292	0.0022	0.1773	33.1%

Gross	1	2	3	4	5	6	7	8	9
f_j	2.5556	1.5283	1.3761	1.2773	1.3170	1.1148	1.0886	1.0648	1.0443
σ_j^2	2,227.06	242.72	235.27	720.66	13,377.69	166.44	35.49	0.78	0.02
u	0.0404								
v^2	42.1016								

Net	1	2	3	4	5	6	7	8	9
f_j	2.5075	1.4858	1.3431	1.2323	1.1744	1.1167	1.1043	1.0588	1.0374
σ_j^2	1,992.25	206.88	36.77	11.43	157.84	32.84	11.97	0.02	0.00
u	0.0546								
v^2	50.2089								

Table 4. Average results with different number of accident years included

No. of Accident Years Included	Gross of Reinsurance					Net of Reinsurance				
	q	$\text{Var}\left(\frac{C_{n+1,n}}{E_{n+1}}\right)$	$\text{Var}(\hat{q})$	$\text{SEP}(\hat{q})$	$\frac{\text{SEP}(\hat{C}_{n+1,n})}{\text{E}(C_{n+1,n})}$	q	$\text{Var}\left(\frac{C_{n+1,n}}{E_{n+1}}\right)$	$\text{Var}(\hat{q})$	$\text{SEP}(\hat{q})$	$\frac{\text{SEP}(\hat{C}_{n+1,n})}{\text{E}(C_{n+1,n})}$
1	50.7%	0.0490	0.0340	0.2852	57.0%	55.5%	0.0295	0.0245	0.2311	41.7%
2	49.9%	0.0485	0.0159	0.2532	51.2%	54.4%	0.0293	0.0110	0.2006	36.9%
3	49.6%	0.0483	0.0112	0.2438	49.4%	54.1%	0.0293	0.0071	0.1906	35.3%
4	49.4%	0.0482	0.0091	0.2394	48.6%	53.9%	0.0292	0.0053	0.1858	34.5%
5	49.4%	0.0482	0.0080	0.2369	48.1%	53.8%	0.0292	0.0042	0.1829	34.1%
6	49.3%	0.0482	0.0072	0.2353	47.8%	53.7%	0.0292	0.0036	0.1810	33.7%
7	49.3%	0.0482	0.0067	0.2342	47.5%	53.6%	0.0292	0.0031	0.1797	33.5%
8	49.3%	0.0481	0.0063	0.2333	47.4%	53.6%	0.0292	0.0027	0.1787	33.3%
9	49.3%	0.0481	0.0060	0.2327	47.3%	53.6%	0.0292	0.0025	0.1779	33.2%
10	49.2%	0.0481	0.0058	0.2322	47.1%	53.6%	0.0292	0.0022	0.1773	33.1%

Table 5. Estimated results using simple average loss ratio

Gross of Reinsurance				Net of Reinsurance			
Accident Year	Premiums	Ultimate Claims	Loss Ratio	Accident Year	Premiums	Ultimate Claims	Loss Ratio
1991	334,566	169,752	50.7%	1991	234,659	130,184	55.5%
$\text{Var}\left(\frac{C_{n+1,n}}{E_{n+1}}\right)$	$\text{Var}(\hat{q}^*)$	$\text{SEP}(\hat{q}^*)$	$\frac{\text{SEP}(\hat{C}_{n+1,n})}{\text{E}(C_{n+1,n})}$	$\text{Var}\left(\frac{C_{n+1,n}}{E_{n+1}}\right)$	$\text{Var}(\hat{q}^*)$	$\text{SEP}(\hat{q}^*)$	$\frac{\text{SEP}(\hat{C}_{n+1,n})}{\text{E}(C_{n+1,n})}$
0.0490	0.0063	0.2353	46.4%	0.0295	0.0027	0.1794	32.3%

Appendix D:

$$\text{Var}(\hat{q}^*) \approx \frac{1}{n^2}\sum_{j=1}^{n-1}\left(\sum_{i=n+1-j}^{n}\frac{\text{E}(C_{i,n})}{E_i f_j}\right)^2 \text{Var}(\hat{f}_j) + \frac{1}{n^2}\sum_{i=1}^{n}\frac{f_{n+1-i}^2 f_{n+2-i}^2 \cdots f_{n-1}^2}{E_i^2}\text{Var}(C_{i,n+1-i}) + \frac{2}{n^2}\sum_{j=1}^{n-1}\sum_{i=1}^{n-j}\left(\sum_{r=n+1-j}^{n}\frac{\text{E}(C_{r,n})}{E_r f_j}\right) \times \left(\frac{f_{n+1-i}f_{n+2-i}\cdots f_{n-1}}{E_i}\right)\text{Cov}(\hat{f}_j, C_{i,n+1-i}), \tag{4.1}$$

which can be estimated by

$$\widehat{\text{Var}}(\hat{q}^*) = \frac{1}{n^2}\sum_{j=1}^{n-1}\left(\sum_{i=n+1-j}^{n}\frac{\hat{C}_{i,n}}{E_i \hat{f}_j}\right)^2 \widehat{\text{Var}}(\hat{f}_j) + \frac{1}{n^2}\sum_{i=1}^{n}\frac{S_{n+1-i,n-1}^2}{E_i^2}\widehat{\text{Var}}(C_{i,n+1-i}) + \frac{2}{n^2}\sum_{j=1}^{n-1}\sum_{i=1}^{n-j}\sum_{r=n+1-j}^{n}\frac{\hat{C}_{r,n}}{E_r \hat{f}_j} \times \frac{S_{n+1-i,n-1}}{E_i}\widehat{\text{Cov}}(\hat{f}_j, C_{i,n+1-i}). \tag{4.2}$$

The estimated results using (3.2), (3.8), and (4.2) are shown in Table 5. The resulting ultimate loss ratios are slightly larger than previously while the standard error of prediction estimates are slightly larger in magnitude but smaller in percentage.

Finally we apply our formulae to some hypothetical data simulated from the compound Poisson model $X_{i,j} = \sum_{k=1}^{N_{i,j}} Y_{i,j,k}$. Let $X_{i,j}$ be a random variable representing the incremental claim amount of accident year i and development year j and so $C_{i,j} = C_{i,j-1} + X_{i,j}$ for $2 \le j \le 10$ and $C_{i,1} = X_{i,1}$. Let $N_{i,j}$ and $Y_{i,j,k}$ be independent random variables representing the number of claims and the size of the kth claim of accident year i and development year j. Let $N_{i,j} \sim \text{Pn}(E_i\lambda_j)$ and $Y_{i,j,k} \sim \text{LN}(\mu,\sigma)$ where λ_j's are equal to 9×10^{-6}, 8×10^{-6}, 7×10^{-6}, 6×10^{-6}, 5×10^{-6}, 5×10^{-6}, 4×10^{-6}, 3×10^{-6}, 2×10^{-6}, 1×10^{-6} respectively for $1 \le j \le 10$, $\mu = 8.8638$, and $\sigma = 0.8326$ ($\text{E}(Y_{i,j,k}) = 10{,}000$ and $\text{SD}(Y_{i,j,k}) = 10{,}000$). We assume E_i grows from 1,000,000 at 10% each year and the unearned premiums are half of E_{11}. Effectively, accident year 11's ultimate loss ratio has a mean of 50% and a variance of 0.0077. We then simulate a run-off triangle based on this compound Poisson model and apply our formulae (3.2), (3.8), and (4.2) to this triangle.

Under the compound Poisson model above, $X_{i,j}$'s are independent while under our model, $C_{i,j+1}$ depends on $C_{i,j}$. Hence we expect our formulae to produce a process error estimate larger than the true variance underlying the simulated data. The simulated run-off triangle and estimated results are presented in Tables 6 and 7.

As expected, the process error estimate of 0.0259 is larger than the underlying variance of 0.0077. In dealing with real claims data, one should check the underlying assumptions thoroughly regarding the conditional relationships or independence between different development years.

Table 6. Simulated data (10% growth)

Claims	1	2	3	4	5	6	7	8	9	10	Premiums
1	80,946	97,396	43,469	40,208	52,068	19,518	14,644	1,692	12,429	1,964	1,000,000
2	63,077	76,181	46,565	68,880	26,412	44,620	53,513	14,540	3,577		1,100,000
3	93,688	112,399	87,149	133,804	17,549	14,814	91,392	35,367			1,210,000
4	116,704	224,930	87,005	61,843	101,357	42,731	27,057				1,331,000
5	192,542	147,366	85,361	61,776	90,964	103,829					1,464,100
6	118,717	97,519	83,964	114,058	89,192						1,610,510
7	156,966	172,695	139,843	156,225							1,771,561
8	175,068	116,656	100,157								1,948,717
9	164,691	112,805									2,143,589
10	239,127										2,357,948
11											1,296,871

Table 7. Estimated results using simple average loss ratio

Accident Year	Premiums	Ultimate Claims	Loss Ratio
11	1,296,871	581,948	44.9%
Var $\left(\frac{C_{n+1,n}}{E_{n+1}}\right)$	Var($\hat{q}^*$)	SEP($\hat{q}^*$)	$\frac{\text{SEP}(\hat{C}_{n+1,n})}{\text{E}(C_{n+1,n})}$
0.0259	0.0030	0.1699	37.9%

5. Concluding remarks

In this paper we examine the weighted and simple average loss ratio estimators and construct a stochastic model to derive some simple approximation formulae to estimate the standard error of prediction for the future claims component of premium liabilities. Based on the idea in Taylor (2000), we deduce the mean square error of prediction as comprising the process error component and the estimation error component, and no covariance term exists as the first part is associated only with the future while the second part only with the past observations. We apply these formulae to some public liability data and simulated data and the results are reasonable in general. Since the starting part of our model follows the structure of the chain ladder method, one may apply the various tests stated in Mack (1994) to check whether the model assumptions are valid for the claims data under investigation.

The formulae derived in this paper appear to serve as a good starting point for assessment of premium liability variability in practice. Nevertheless, there are other practical considerations in dealing with premium liabilities such as the insurance cycle, claims development in the tail, catastrophes, superimposed inflation, multi-year policies, policy administration and claims handling expenses, future recoveries, future reinsurance costs, retrospectively rated policies, unclosed business, refund claims, and future changes in reinsurance, claims management, and underwriting. To deal with these issues, a practitioner needs to judgmentally adjust the data or make an explicit allowance, based on managerial, internal, and industry information.

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Appendix A

In this appendix we prove that the estimators $\hat{u}$ and $\hat{v}^2$ of (2.8) and (2.9) are unbiased:

$$\mathrm{E}(\hat{u}) = \mathrm{E}\left(\frac{\sum_{r=1}^{n} C_{r,1}}{\sum_{r=1}^{n} E_r}\right) = \frac{\sum_{r=1}^{n} \mathrm{E}(C_{r,1})}{\sum_{r=1}^{n} E_r} = \frac{\sum_{r=1}^{n} E_r u}{\sum_{r=1}^{n} E_r} = u; \qquad \text{(from (2.4))}$$

$$\mathrm{Var}(\hat{u}) = \mathrm{Var}\left(\frac{\sum_{r=1}^{n} C_{r,1}}{\sum_{r=1}^{n} E_r}\right) = \frac{\sum_{r=1}^{n} \mathrm{Var}(C_{r,1})}{\left(\sum_{r=1}^{n} E_r\right)^2} = \frac{\sum_{r=1}^{n} E_r v^2}{\left(\sum_{r=1}^{n} E_r\right)^2} = \frac{v^2}{\sum_{r=1}^{n} E_r}; \qquad \text{(from (2.3) and (2.5))}$$

$$\mathrm{E}(\hat{v}^2) = \mathrm{E}\left[\frac{1}{n-1}\sum_{r=1}^{n} E_r \left(\frac{C_{r,1}}{E_r} - \hat{u}\right)^2\right] = \frac{1}{n-1}\sum_{r=1}^{n} E_r \mathrm{E}\left[\left(\frac{C_{r,1}}{E_r} - \hat{u}\right)^2\right]$$

$$= \frac{1}{n-1}\sum_{r=1}^{n} E_r \left\{\mathrm{E}\left(\frac{C_{r,1}^2}{E_r^2}\right) - 2\mathrm{E}\left(\frac{C_{r,1}}{E_r}\frac{\sum_{g=1}^{n} C_{g,1}}{\sum_{g=1}^{n} E_g}\right) + \mathrm{E}(\hat{u}^2)\right\}$$

$$= \frac{1}{n-1}\sum_{r=1}^{n} E_r \left\{\mathrm{Var}\left(\frac{C_{r,1}}{E_r}\right) + \mathrm{E}\left(\frac{C_{r,1}}{E_r}\right)^2 - \frac{2}{E_r}\left[\frac{\sum_{g=1}^{n} \mathrm{E}(C_{r,1})\mathrm{E}(C_{g,1})}{\sum_{g=1}^{n} E_g} + \frac{\mathrm{Var}(C_{r,1})}{\sum_{g=1}^{n} E_g}\right] + \mathrm{Var}(\hat{u}) + \mathrm{E}(\hat{u})^2\right\}$$

(from (2.3))

$$= \frac{1}{n-1}\sum_{r=1}^{n} E_r \left\{\frac{v^2}{E_r} + u^2 - \frac{2}{E_r}\left[\frac{\sum_{g=1}^{n} E_r E_g u^2}{\sum_{g=1}^{n} E_g} + \frac{E_r v^2}{\sum_{g=1}^{n} E_g}\right] + \frac{v^2}{\sum_{g=1}^{n} E_g} + u^2\right\}$$

(from (2.4), (2.5), and above)

$$= \frac{1}{n-1}\sum_{r=1}^{n}\left(v^2 - \frac{E_r v^2}{\sum_{g=1}^{n} E_g}\right) = v^2.$$

Appendix B

We prove in this appendix that both (3.1) and (3.2) give unbiased estimators. To start with, we have to show the following with repeated use of the law of total expectation:

$$\mathrm{E}\left(\frac{C_{n+1,n}}{E_{n+1}}\right) = \mathrm{E}\left(\mathrm{E}\left(\frac{C_{n+1,n}}{E_{n+1}} \,\middle|\, C_{n+1,1}, C_{n+1,2}, \ldots, C_{n+1,n-1}\right)\right) = \mathrm{E}\left(\frac{C_{n+1,n-1}}{E_{n+1}} f_{n-1}\right) \qquad \text{(from (2.1))}$$

$$= \mathrm{E}\left(\mathrm{E}\left(\frac{C_{n+1,n-1}}{E_{n+1}} f_{n-1} \,\middle|\, C_{n+1,1}, C_{n+1,2}, \ldots, C_{n+1,n-2}\right)\right) = \mathrm{E}\left(\frac{C_{n+1,n-2}}{E_{n+1}} f_{n-2} f_{n-1}\right)$$

(from (2.1) again)

$$= \cdots = \mathrm{E}\left(\frac{C_{n+1,1}}{E_{n+1}} f_1 f_2 \ldots f_{n-1}\right) \qquad \text{(repeat above)}$$

$$= u f_1 f_2 \ldots f_{n-1}. \qquad \text{(from (2.4))}$$

The above results can readily be extended to accident years and development years other than $n+1$ and n shown here.

Mack (1993) proves that $\hat{f}_j$ is unbiased and that $\hat{f}_j$ and $\hat{f}_h$ are uncorrelated for $j \neq h$. First we look at the expected value of the weighted average estimator of (3.1):

$$\begin{aligned}
\mathrm{E}(\hat{q}) &= \mathrm{E}\left(\frac{\sum_{i=1}^{n} C_{i,n+1-i}\hat{f}_{n+1-i}\hat{f}_{n+2-i}\cdots\hat{f}_{n-1}}{\sum_{i=1}^{n} E_i}\right) \\
&= \frac{\sum_{i=1}^{n} \mathrm{E}(C_{i,n+1-i}) f_{n+1-i} f_{n+2-i}\cdots f_{n-1}}{\sum_{i=1}^{n} E_i} && \text{(from Mack (1993))} \\
&= \frac{\sum_{i=1}^{n} E_i u f_1 f_2 \ldots f_{n-1}}{\sum_{i=1}^{n} E_i} = u f_1 f_2 \ldots f_{n-1} = q. && \text{(from above)}
\end{aligned}$$

Similarly, the expected value of the simple average estimator of (3.2) is as follows:

$$\begin{aligned}
\mathrm{E}(\hat{q}^*) &= \mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{n} \frac{C_{i,n+1-i}\hat{f}_{n+1-i}\hat{f}_{n+2-i}\cdots\hat{f}_{n-1}}{E_i}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}\left(\frac{C_{i,n+1-i}}{E_i}\right) f_{n+1-i} f_{n+2-i}\cdots f_{n-1} && \text{(from Mack (1993))} \\
&= \frac{1}{n}\sum_{i=1}^{n} u f_1 f_2 \ldots f_{n-1} = u f_1 f_2 \ldots f_{n-1} = q. && \text{(from above)}
\end{aligned}$$

Appendix C

We derive the process error component in (3.7) of the mean square error of prediction as follows, with repeated use of the law of total variance:

$$\begin{aligned}
\mathrm{Var}\left(\frac{C_{n+1,n}}{E_{n+1}}\right) &= \mathrm{E}\left(\mathrm{Var}\left(\frac{C_{n+1,n}}{E_{n+1}} \,\middle|\, C_{n+1,1}, C_{n+1,2}, \ldots, C_{n+1,n-1}\right)\right) + \mathrm{Var}\left(\mathrm{E}\left(\frac{C_{n+1,n}}{E_{n+1}} \,\middle|\, C_{n+1,1}, C_{n+1,2}, \ldots, C_{n+1,n-1}\right)\right) \\
&= \mathrm{E}\left(\frac{C_{n+1,n-1}}{E_{n+1}^2}\sigma_{n-1}^2\right) + \mathrm{Var}\left(\frac{C_{n+1,n-1}}{E_{n+1}} f_{n-1}\right) && \text{(from (2.2) and (2.1))} \\
&= \mathrm{E}\left(\frac{C_{n+1,n-1}}{E_{n+1}^2}\sigma_{n-1}^2\right) + \mathrm{E}\left(\mathrm{Var}\left(\frac{C_{n+1,n-1}}{E_{n+1}} f_{n-1} \,\middle|\, C_{n+1,1}, C_{n+1,2}, \ldots, C_{n+1,n-2}\right)\right) \\
&\quad + \mathrm{Var}\left(\mathrm{E}\left(\frac{C_{n+1,n-1}}{E_{n+1}} f_{n-1} \,\middle|\, C_{n+1,1}, C_{n+1,2}, \ldots, C_{n+1,n-2}\right)\right) \\
&= \mathrm{E}\left(\frac{C_{n+1,n-1}}{E_{n+1}^2}\sigma_{n-1}^2\right) + \mathrm{E}\left(\frac{C_{n+1,n-2}}{E_{n+1}^2}\sigma_{n-2}^2 f_{n-1}^2\right) + \mathrm{Var}\left(\frac{C_{n+1,n-2}}{E_{n+1}} f_{n-2} f_{n-1}\right) \\
& && \text{(from (2.2) and (2.1) again)} \\
&= \cdots = \mathrm{E}\left(\frac{C_{n+1,n-1}}{E_{n+1}^2}\sigma_{n-1}^2\right) + \mathrm{E}\left(\frac{C_{n+1,n-2}}{E_{n+1}^2}\sigma_{n-2}^2 f_{n-1}^2\right) + \cdots + \mathrm{E}\left(\frac{C_{n+1,1}}{E_{n+1}^2}\sigma_1^2 f_2^2 f_3^2 \ldots f_{n-1}^2\right) \\
&\quad + \mathrm{Var}\left(\frac{C_{n+1,1}}{E_{n+1}} f_1 f_2 \ldots f_{n-1}\right) && \text{(repeat above)} \\
&= \frac{1}{E_{n+1}}\left[\mathrm{E}\left(\frac{C_{n+1,n-1}}{E_{n+1}}\right)\sigma_{n-1}^2 + \mathrm{E}\left(\frac{C_{n+1,n-2}}{E_{n+1}}\right)\sigma_{n-2}^2 f_{n-1}^2 + \cdots + \mathrm{E}\left(\frac{C_{n+1,1}}{E_{n+1}}\right)\sigma_1^2 f_2^2 f_3^2 \ldots f_{n-1}^2\right] \\
&\quad + \frac{v^2}{E_{n+1}} f_1^2 f_2^2 \ldots f_{n-1}^2 && \text{(from (2.5))}
\end{aligned}$$

$$= \frac{1}{E_{n+1}}\left[\mathrm{E}\left(\frac{C_{n+1,n}}{E_{n+1}}\right)\frac{\sigma_{n-1}^2}{f_{n-1}} + \mathrm{E}\left(\frac{C_{n+1,n}}{E_{n+1}}\right)\frac{\sigma_{n-2}^2}{f_{n-2}}f_{n-1} + \cdots + \mathrm{E}\left(\frac{C_{n+1,n}}{E_{n+1}}\right)\frac{\sigma_1^2}{f_1}f_2f_3\ldots f_{n-1}\right]$$

$$+ \frac{v^2}{E_{n+1}}f_1^2f_2^2\ldots f_{n-1}^2 \qquad \text{(from Appendix B)}$$

$$= \frac{1}{E_{n+1}}\mathrm{E}\left(\frac{C_{n+1,n}}{E_{n+1}}\right)\sum_{j=1}^{n-1}\frac{\sigma_j^2}{f_j}f_{j+1}f_{j+2}\ldots f_{n-1} + \frac{v^2}{E_{n+1}}f_1^2f_2^2\ldots f_{n-1}^2.$$

Appendix D

In the following we derive the estimation error component in (3.9) of the mean square error of prediction. We first apply the Taylor series expansion to the estimator $\hat{q}$ of (3.1):

$$\hat{q} = \frac{\sum_{i=1}^n C_{i,n+1-i}\hat{f}_{n+1-i}\hat{f}_{n+2-i}\cdots\hat{f}_{n-1}}{\sum_{i=1}^n E_i} \approx \frac{\sum_{i=1}^n \mathrm{E}(C_{i,n+1-i})f_{n+1-i}f_{n+2-i}\cdots f_{n-1}}{\sum_{i=1}^n E_i}$$

$$+ \frac{1}{\sum_{i=1}^n E_i}\sum_{j=1}^{n-1}(\hat{f}_j - f_j)\sum_{i=1}^n \frac{\partial}{\partial \hat{f}_j}C_{i,n+1-i}\hat{f}_{n+1-i}\hat{f}_{n+2-i}\cdots\hat{f}_{n-1}\bigg|_{C_{i,n+1-i}=\mathrm{E}(C_{i,n+1-i});\,\hat{f}_j=f_j}$$

$$+ \frac{1}{\sum_{i=1}^n E_i}\sum_{i=1}^{n}(C_{i,n+1-i} - \mathrm{E}(C_{i,n+1-i}))\sum_{r=1}^n \frac{\partial}{\partial C_{i,n+1-i}}C_{r,n+1-r}\hat{f}_{n+1-r}\hat{f}_{n+2-r}\cdots\hat{f}_{n-1}\bigg|_{C_{i,n+1-i}=\mathrm{E}(C_{i,n+1-i});\,\hat{f}_j=f_j}$$

$$= \mathrm{E}(\hat{q}) + \frac{1}{\sum_{i=1}^n E_i}\sum_{j=1}^{n-1}(\hat{f}_j - f_j)\sum_{i=n+1-j}^{n}\frac{\mathrm{E}(C_{i,n})}{f_j} \qquad \text{(from Appendix B)}$$

$$+ \frac{1}{\sum_{i=1}^n E_i}\sum_{i=1}^{n}(C_{i,n+1-i} - \mathrm{E}(C_{i,n+1-i}))f_{n+1-i}f_{n+2-i}\cdots f_{n-1}.$$

Moving the term $\mathrm{E}(\hat{q})$ to the left-hand side of the equation and then taking the expectation on the square of the resulting equation, we deduce that:

$$\mathrm{Var}(\hat{q}) \approx \frac{1}{\left(\sum_{i=1}^n E_i\right)^2}\mathrm{E}\left[\left(\sum_{j=1}^{n-1}(\hat{f}_j - f_j)\sum_{i=n+1-j}^{n}\frac{\mathrm{E}(C_{i,n})}{f_j}\right)^2\right]$$

$$+ \frac{1}{\left(\sum_{i=1}^n E_i\right)^2}\mathrm{E}\left[\left(\sum_{i=1}^{n}(C_{i,n+1-i} - \mathrm{E}(C_{i,n+1-i}))f_{n+1-i}f_{n+2-i}\cdots f_{n-1}\right)^2\right]$$

$$+ \frac{2}{\left(\sum_{i=1}^n E_i\right)^2}\mathrm{E}\left[\left(\sum_{j=1}^{n-1}(\hat{f}_j - f_j)\sum_{i=n+1-j}^{n}\frac{\mathrm{E}(C_{i,n})}{f_j}\right)\left(\sum_{i=1}^{n}(C_{i,n+1-i} - \mathrm{E}(C_{i,n+1-i}))f_{n+1-i}f_{n+2-i}\cdots f_{n-1}\right)\right]$$

$$= \frac{1}{\left(\sum_{i=1}^n E_i\right)^2}\sum_{j=1}^{n-1}\left(\sum_{i=n+1-j}^{n}\frac{\mathrm{E}(C_{i,n})}{f_j}\right)^2\mathrm{Var}(\hat{f}_j) \qquad (\hat{f}_j\text{'s are unbiased and uncorrelated})$$

$$+ \frac{1}{\left(\sum_{i=1}^n E_i\right)^2}\sum_{i=1}^{n}f_{n+1-i}^2f_{n+2-i}^2\cdots f_{n-1}^2\,\mathrm{Var}(C_{i,n+1-i}) \qquad \text{(from (2.3))}$$

$$+ \frac{2}{\left(\sum_{i=1}^n E_i\right)^2}\sum_{j=1}^{n-1}\sum_{i=1}^{n-j}\left(\sum_{r=n+1-j}^{n}\frac{\mathrm{E}(C_{r,n})}{f_j}\right)(f_{n+1-i}f_{n+2-i}\cdots f_{n-1})\mathrm{Cov}(\hat{f}_j, C_{i,n+1-i}).$$

($\hat{f}_j$ is unbiased; $\hat{f}_j$ and $C_{i,n+1-i}$ are independent for $j > n - i$ due to (2.3)).

As shown in Mack (1993), $\mathrm{E}(\hat{f}_j \mid B_j) = f_j$ and $\mathrm{Var}(\hat{f}_j \mid B_j) = \sigma_j^2 / \sum_{r=1}^{n-j} C_{r,j}$ where B_j represents all the past claims data to development year j. We then deduce the following:

$$
\begin{aligned}
\mathrm{Var}(\hat{f}_j) &= \mathrm{E}(\mathrm{Var}(\hat{f}_j \mid B_j)) + \mathrm{Var}(\mathrm{E}(\hat{f}_j \mid B_j)) \\
&= \mathrm{E}\left(\frac{\sigma_j^2}{\sum_{r=1}^{n-j} C_{r,j}}\right) + \mathrm{Var}(f_j) \\
&= \mathrm{E}\left(\frac{\sigma_j^2}{\sum_{r=1}^{n-j} C_{r,j}}\right),
\end{aligned}
$$

which can be approximated by $\widehat{\mathrm{Var}}(\hat{f}_j) = \hat{\sigma}_j^2 / \sum_{r=1}^{n-j} C_{r,j}$.

Repeatedly using the law of total variance as in Appendix C, we derive that $\mathrm{Var}(C_{i,n+1-i}) = \mathrm{E}(C_{i,n+1-i}) \cdot \sum_{j=1}^{n-i} (\sigma_j^2 / f_j) f_{j+1} f_{j+2} \ldots f_{n-i} + E_i v^2 f_1^2 \cdot f_2^2 \ldots f_{n-i}^2$, which can then be estimated by $\widehat{\mathrm{Var}}(C_{i,n+1-i}) = C_{i,n+1-i} \sum_{j=1}^{n-i} (\hat{\sigma}_j^2 / \hat{f}_j) S_{j+1,n-i} + E_i \hat{v}^2 \cdot S_{1,n-i}^2$.

Finally, we derive the covariance between $\hat{f}_j$ and $C_{i,n+1-i}$ for $j \le n-i$ as follows:

$$
\begin{aligned}
\mathrm{Cov}(\hat{f}_j, C_{i,n+1-i}) &= \mathrm{E}(\hat{f}_j C_{i,n+1-i}) - \mathrm{E}(C_{i,n+1-i}) f_j && (\hat{f}_j \text{ is unbiased}) \\
&= \mathrm{E}(\mathrm{E}(\hat{f}_j C_{i,n+1-i} \mid B_{n-i})) - \mathrm{E}(C_{i,n+1-i}) f_j \\
&= \mathrm{E}(\hat{f}_j C_{i,n-i} f_{n-i}) - \mathrm{E}(C_{i,n+1-i}) f_j && (\text{from } (2.1)) \\
&= \cdots = \mathrm{E}(\hat{f}_j C_{i,j+1} f_{j+1} f_{j+2} \ldots f_{n-i}) - \mathrm{E}(C_{i,n+1-i}) f_j && (\text{repeat above}) \\
&= \mathrm{E}\left(\frac{\sum_{r=1}^{n-j} C_{r,j+1} C_{i,j+1}}{\sum_{r=1}^{n-j} C_{r,j}} f_{j+1} f_{j+2} \ldots f_{n-i}\right) - \mathrm{E}(C_{i,n+1-i}) f_j && (\text{from } (2.6)) \\
&= \mathrm{E}\left(\frac{\sum_{r=1}^{n-j} \mathrm{E}(C_{r,j+1} C_{i,j+1} \mid B_j)}{\sum_{r=1}^{n-j} C_{r,j}} f_{j+1} f_{j+2} \ldots f_{n-i}\right) - \mathrm{E}(C_{i,n+1-i}) f_j \\
&= \mathrm{E}\left(\frac{\sum_{r=1}^{n-j} \mathrm{E}(C_{r,j+1} \mid B_j) \mathrm{E}(C_{i,j+1} \mid B_j) + \mathrm{Var}(C_{i,j+1} \mid B_j)}{\sum_{r=1}^{n-j} C_{r,j}} f_{j+1} f_{j+2} \ldots f_{n-i}\right) - \mathrm{E}(C_{i,n+1-i}) f_j \\
& && (\text{from } (2.3)) \\
&= \mathrm{E}\left(\frac{\sum_{r=1}^{n-j} C_{r,j} C_{i,j} f_j^2 + C_{i,j} \sigma_j^2}{\sum_{r=1}^{n-j} C_{r,j}} f_{j+1} f_{j+2} \ldots f_{n-i}\right) - \mathrm{E}(C_{i,n+1-i}) f_j && (\text{from } (2.1) \text{ and } (2.2)) \\
&= \mathrm{E}\left(C_{i,j} f_j^2 f_{j+1} f_{j+2} \ldots f_{n-i} + \frac{C_{i,j} \sigma_j^2}{\sum_{r=1}^{n-j} C_{r,j}} f_{j+1} f_{j+2} \ldots f_{n-i}\right) - \mathrm{E}(C_{i,n+1-i}) f_j \\
&= \mathrm{E}\left(\frac{C_{i,n+1-i}}{\sum_{r=1}^{n-j} C_{r,j}}\right) \frac{\sigma_j^2}{f_j}, && (\text{from Appendix B})
\end{aligned}
$$

which can be approximated by $\widehat{\mathrm{Cov}}(\hat{f}_j, C_{i,n+1-i}) = (C_{i,n+1-i} / \sum_{r=1}^{n-j} C_{r,j})(\hat{\sigma}_j^2 / \hat{f}_j)$.

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