TMV-Based Capital Allocation for Multivariate Risks

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ABSTRACT

This paper studies a novel capital allocation framework based on the tail mean-variance (TMV) principle for multivariate risks. The new capital allocation model has many intriguing properties, such as controlling the magnitude and variability of tail risks simultaneously. General formulas for optimal capital allocations are discussed according to the semideviation distance measure. In particular, we discuss the optimal capital allocation for comonotonic risks, and risks from multivariate elliptical distribution and multivariate skew-t distribution. Some numerical examples are given to illustrate the results, and real data from an insurance company is analyzed as well.

KEYWORDS

Capital allocation, Lagrange multiplier, mean-variance, skew-t distribution, tail risks
1. Introduction and motivation

In the actuarial literature, a fundamental question is how to allocate the total amount of risk capital to different subportfolios, divisions, or lines of businesses. The allocation problem is very important since the amount of risk capital allocated to a business consisting of multiple lines of businesses is typically less than the sum of amounts of risk capital that would need to be withheld for each business separately. Heterogeneity and dependence that may exist between the performances of various business units make capital allocation a nontrivial exercise. Therefore, there exists an extensive amount of literature on this subject with a number of proposed capital allocation algorithms. For example, Myers and Read (2001) considered capital allocation principles based on the marginal contribution of each business unit to the company’s default option. Denault (2001) discussed capital allocations from the perspective of game theory. The first multivariate top-down model considered in Panjer (2002) studies the particular case of multivariate, normally distributed risks and provides an explicit expression of marginal cost-based allocations using TVaR (tail value-at-risk) risk measure. This work has been extended by Landsman and Valdez (2003) to model risks using multivariate elliptical distributions, which include the multivariate normal as a special case; see also Dhaene et al. (2008). Furman and Landsman (2005) studied the capital allocation for the risks following multivariate gamma distributions. Cossette et al. (2013) discussed the multivariate risks with mixed Erlang marginals and the dependence structure is modeled by the Farlie-Gumbel-Morgenstern copula. One may refer to Dhaene et al. (2012), Xu and Hu (2012), Tsanakas (2004), and Furman and Zitikis (2008) and references therein for the recent developments on this topic.

Assume that a firm has a portfolio of risks $X_1, \ldots, X_n$, and wishes to allocate the total capital $K = k_1 + \ldots + k_n$ to the corresponding risks. The total risk is then

$$S = X_1 + \ldots + X_n.$$ 

Recently, Dhaene et al. (2012) proposed a criterion to set the capital amount $k_i$ to $X_i$ as close as possible to minimize the loss. Specifically, the criterion is to minimize the following loss function

$$L(k) = \sum_{i=1}^n D(X_i - k_i),$$  \hspace{1cm} (1.1)

where $D$ is some suitable distance measurement function, and $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$. A lot of work has been motivated by this criterion; see Xu and Hu (2012), Zaks (2013), Cheung, Rong, and Yam (2014), and others. In fact, the idea of minimizing the loss function has been discussed in the framework of premium calculation. For example, Zaks et al. (2006) used quadratic distance measure $D(x) = x^2$, and Laeven and Goovaerts (2004) used the semi-deviation function $D(x) = \max\{x, 0\}$ as distance measure. This topic was further pursued in Frostig, Zaks and Levikson (2007), where they used the general convex distance measure. However, most of the discussion on capital allocations in the literature has focused only on the magnitude of the loss function $L$. In practice, the variability also plays an essential role in determining the capital allocations. Indeed, the relevant idea has already appeared in the premium calculation. Furman and Landsman (2006) used the tail variance risk measure to estimate the variability along the tails, and to compute the premium based on the tail variance premium (TVP) model

$$\text{TVP}_q(X) = \text{TCE}_q(X) + \beta \text{TV}_q(X), \hspace{1cm} \beta \geq 0,$$

where $\text{TCE}_q$ and $\text{TV}_q$ represent tail conditional expectation, and tail conditional variance, respectively. That is,

$$\text{TCE}_q(X) = \mathbb{E}(X|X > x_q),$$

$$\text{TV}_q(X) = \text{Var}(X|X > x_q),$$

where $x_q$ is $q$th quantile of risk $X$. See also Landsman, Pat, and Dhaene (2013) for the discussion on the tail variance related premium calculation. Motivated by this observation, Xu and Mao (2013) proposed a TMV model to discuss the optimal capital allocations, where they defined the loss function as
where $\text{VaR}_q(S)$ is the $q$th quantile of $S$, and considered the following function

\[
\pi[G(X; k, q)] = \mathbb{E}[G(X; k, q)] + \beta \text{Var}(G(X; k, q)),
\]

where $\pi(\cdot)$ is the mean-variance risk measurement, which has been widely used in practice (Laeven and Goovaerts 2004). The TMV model has many intriguing properties, such as controlling the magnitude and variability of tail loss, and providing neat optimal allocation formulas. From the economic perspective, the perfect case is that the company could prepare the capital to match the loss exactly, since too much or less capital would result in the loss of revenue for a company. Therefore, a company should prefer a capital allocation rule which could provide the capital to match the loss as close as possible. It is apparent that controlling the magnitude of deviation of the capital from the loss is important. However, the variability of deviation is also essential in determining the required capital, as the larger variability would lead to more risk for the company. Therefore, the property of controlling the magnitude and variability is appealing in determining the required capital for business lines.

In practice, however, the shortage of capital may often result in much severer consequences than that caused by the excess of capital (Myers and Read 2001; Erel, Myers and Read 2013), which suggests that the semideviation function may be preferred in practice as the distance measure. This issue is also related to the capital allocation of homeland security, an area that has become centrally important since the terrorist attacks of September 11, 2001. Since catastrophes are highly risky and could lead to severe consequences, the Department of Homeland Security (DHS) has endeavored to use risk management to determine the capital allocations on prevention, response, and recovery from such national catastrophes. The budget in DHS is allocated via the program called UASI (Urban Area Security Initiative) each year. For example, DHS allocated a total of $490.4 million in 2012, $558.7 million in 2013, and $587.0 million in 2014 to urban areas to prevent terrorist attacks. The effective allocation of the total capital to urban areas is an important but challenging problem, which has received much attention in the security area (cf. Hu, Homem-de-Mello, and Mehrotra 2011). A popular distance measure used in this area is the semideviation function. In fact, in the literature of actuary science, the semideviation function has been widely used in the stop-loss premium calculation (Dhaene et al. 2012).

Based on the above discussion, in this paper, we are motivated to study the capital allocation based on the TMV model with the loss function defined as

\[
L(X; k, q) = \sum_{i=1}^{n} [(X_i - k_i)_+ | S > \text{VaR}_q(S)].
\]

We consider the following general mean-variance model,

\[
\begin{align*}
\min_{k \in A} & \quad \pi[L(X; k, q)]; \\
\text{s.t.} & \quad A = \{k \in \mathbb{R}^n : \sum_{i=1}^{n} k_i = K, i = 1, \ldots, n\}.
\end{align*}
\]

(1.2)

where $\pi(\cdot)$ is the mean-variance risk measurement, and $\beta \geq 0$. It is worth pointing out that Laeven and Goovaerts (2004) considered a special case of the TMV model (1.2). They discussed the case of $n = 2$ but without considering the tail risks. Specifically, they discussed the optimal capital allocation based on minimizing the following loss function,

\[
\pi[L(X; k)] = \mathbb{E}[(X_1 - k_1)_+ + (X_2 - k_2)_+] + \beta \text{Var}[(X_1 - k_1)_+ + (X_2 - k_2)_+],
\]

over $k \in A$. Therefore, the TMV model (1.2) is a natural extension of their model.

Our main contributions in this paper are summarized as follows. First, we derive the general equations
for the TMV model (1.2), based on which the numerical programming could be easily implemented. Second, we discuss the special case of comonotonic risks, and the closed-form solutions are obtained. Third, we compute the key quantities of optimal capital allocation formulas for multivariate elliptical distributions, and Monte Carlo simulation for those quantities of multivariate skew-t distributions is also mentioned. Finally, we conduct a real data analysis and discuss the optimal capital allocation based on the new model.

The rest of the paper is organized as follows. In Section 2, we derive the general equations for the TMV model, and discuss a special case. Section 3 studies the optimal capital allocations for the comonotonic risks. In Section 4, we present some numerical examples to illustrate the different factors that affect the capital allocations and conduct a real data analysis of capital allocations for an insurance company. In the last section, we summarize the results and present some discussion.

2. Optimal capital allocation: General results

In this section, we provide general capital allocation equations for the TMV model (1.2). To facilitate the discussion, let us denote the conditional survival function of \( [X_i \mid S > \text{VaR}_q(S)] \) by

\[
F_{i,S}(k_i) = \mathbb{P}(X_i > k_i \mid S > \text{VaR}_q(S)), \quad i = 1, \ldots, n.
\]

The conditional expectation of risk excess \( [(X_i - k_i) + \mid S > \text{VaR}_q(S)] \) is denoted by

\[
\text{ECT}_{i,S}(k_i) = \mathbb{E}[ (X_i - k_i) + \mid S > \text{VaR}_q(S)],
\]

and the covariance between \( [(X_i - k_i) + \mid S > \text{VaR}_q(S)] \) and \( [I(X_j \geq k_j) \mid S > \text{VaR}_q(S)] \) is represented by

\[
\text{Cov}_{S,i}(k_i, k_j) = \text{Cov}[(X_i - k_i), I(X_j \geq k_j) \mid S > \text{VaR}_q(S)],
\]

for \( i, j = 1, \ldots, n \), where \( I(\cdot) \) is the indicator function.

In the following, by using the methodology of Lagrange multipliers, we present the optimal capital allocation equations based on the TMV model (1.2), and the uniqueness condition is also given. The proof is moved to the Appendix for the sake of readability.

**Theorem 2.1.** For the TMV model (1.2), assume that \( X_1, \ldots, X_n \) are continuous risks, then an optimal allocation solution \( k^* = (k^*_1, \ldots, k^*_n) \) is given by the following equations, for any \( l = 1, 2, \ldots, n \),

\[
F_{i,S}(k^*_i) + 2\beta \sum_{j=1}^n \text{Cov}_{S,i}(k^*_i, k^*_j)
= F_{i,S}(k^*_i) + 2\beta \sum_{j=1}^n \text{Cov}_{S,i}(k^*_j, k^*_i) \quad (2.1)
\]

and

\[
k^*_1 + \ldots + k^*_n = K.
\]

Further, if, for any \( l = 1, 2, \ldots, n \),

\[
1 + 2\beta \sum_{j=1}^n \mathbb{E}[ (X_j - k^*_j) \mid X_l = k^*_l, S > \text{VaR}_q(S)] > 2\beta \sum_{j=1}^n \text{ECT}_{i,S}(k^*_j) \quad (2.2)
\]

then the solution is unique.

From Theorem 2.1, it is seen that the capital allocations based on Model (1.2) depend not only on the magnitude of tail risks but also the covariance among the tail risks. This property would allow the company to control the tail risks from both the magnitude and variability perspectives. In general, there does not exist an analytical solution to Eq. (2.1). The key quantities required to solve the equation are \( \text{ECT}_{i,S}(\cdot) \), \( \text{Cov}_{S,i}(\cdot, \cdot) \), which, however, could be efficiently computed by using any computer software; see Section 4 for examples. It can be seen from Eq. (2.2) that when \( \beta \) is small, then the uniqueness condition is easily satisfied. In the following, we discuss a special case of \( \beta = 0 \) for Theorem 2.1, i.e., without considering the penalty on the tail variance. For this case, a closed-form solution could be obtained.
3. Comonotonic risks

Comonotonicity, an extremal form of positive dependence, has been widely used in finance and actuarial science over the last two decades. It is well known that the comonotonic random variables are always moving in the same direction simultaneously and hence are considered as extreme dependent risks. Refer to Dhaene et al. (2002a; 2002b) for the properties and applications of this concept in actuarial science and finance. For a company with several business lines, it is particularly important for them to prepare for the worst scenario. It is known in the literature that the aggregate risk of comonotonic risks with finite means may be regarded as the most dangerous case in terms of convex order (Dhaene et al. 2002a). From the perspective of capital allocation allocations, it would be interesting to know whether the comonotonic dependence structure among risks is the most dangerous case in terms of some stochastic measure. Further, if it is the most dangerous scenario, what is the optimal capital allocation strategy? In this section, we first show that the comonotonic risks are the most dangerous risks for the capital allocations in the sense that the expected tail loss is the largest. Then, we discuss the optimal capital allocation based on the TMV model (1.2). We need the following two lemmas.

**Lemma 3.1.** A random vector \((X_1, \ldots, X_n)\) is comonotonic if and only if there are increasing real-valued functions \(f_1, \ldots, f_n\) and a random variable \(W\) such that
\[
(X_1, \ldots, X_n) \overset{d}{=} (f_1(W), \ldots, f_n(W)),
\]
where \(\overset{d}{=}\) represents that both sides of equality have the same distribution.

The following lemma, essentially due to Sordo et al. (2013), will also be used in the sequel.
**Lemma 3.2.** Let $X$ and $Y$ be two continuous risks with strictly increasing distribution functions $F$ and $G$, respectively. Then, for $q \in (0, 1]$, it holds that

$$[X|Y > G^{-1}(q)] \leq_{s} [X|X > F^{-1}(q)],$$

where $\leq_{s}$ represents the usual stochastic order (Shaked and Shanthikumar 2007). Particularly if $X$ and $Y$ are comonotonic, then

$$[X|Y > G^{-1}(q)] = [X|X > F^{-1}(q)].$$

By utilizing the above two lemmas, we show that the comonotonic risks result in the largest tail losses, which may have their own interest.

**Theorem 3.3.** Let $(X_1, \ldots, X_n)$ be a continuous random vector with strictly increasing distribution functions, and $(X'_1, \ldots, X'_n)$ represents its comonotonic counterpart. Then,

$$\mathbb{E} \left[ \sum_{i=1}^{n} (X_i - k_i)_+ | S > \text{VaR}_q(S) \right] \leq \mathbb{E} \left[ \sum_{i=1}^{n} (X'_i - k'_i)_+ | S > \text{VaR}_q(S') \right].$$

**Proof:** Since $(X'_i, S')$ are comonotonic for $i = 1, \ldots, n$, from Lemma 3.1 it follows that

$$((X'_i - k'_i)_+, S')$$

are also comonotonic, since $h(x) = (x - k_i)_+$ is an increasing function of $x$. Therefore, according to Lemma 3.2, we have

$$[(X'_i - k'_i)_+, S' > \text{VaR}_q(S')] \leq [(X'_i - k'_i)_+, (X'_i - k'_i)_+ > \text{VaR}_q((X'_i - k'_i)_+)] \leq [(X_i - k_i)_+, (X_i - k_i)_+ > \text{VaR}_q((X_i - k_i)_+)] \geq_{s} [(X_i - k_i)_+, S > \text{VaR}_q(S)].$$

Hence, the required result follows immediately. ■

Now, let us discuss the optimal capital allocation based on Model (1.2) for this worst scenario, i.e., $X_1, \ldots, X_n$ are comonotonic risks.

**Theorem 3.4.** Under Model (1.2), a unique optimal allocation solution $k^*_n = (k^*_1, \ldots, k^*_n)$ when $(X_1, \ldots, X_n)$ are comonotonic risks with strictly increasing distributions is given by

$$k^*_n = F_{1:n}^{-1}(F_{X_n}(K)), \quad i = 1, \ldots, n, \quad (3.1)$$

where $S^c = \sum_{i=1}^{n} F_{i:n}^{-1}(U)$, with $F_{i:n}^{-1}(U) = [X|S > \text{VaR}_q(S)]$ almost surely.

**Proof:** Note that

$$\sum_{j=1}^{n} \text{Cov}_{j,S}(k^*_j, k^*_j)$$

$$= \sum_{j=1}^{n} \text{Cov} \left[ (X_j - k^*_j)_+, I(X_j \geq k^*_j) | S > \text{VaR}_q(S) \right]$$

$$= \text{Cov} \left[ \sum_{j=1}^{n} (X_j - k^*_j)_+, I(X_j \geq k^*_j) | S > \text{VaR}_q(S) \right].$$

Since $(X_1, \ldots, X_n)$ is a comonotonic vector, it holds that

$$[(X_1, \ldots, X_n) | S > \text{VaR}_q(S)]$$

is also comonotonic, and, further,

$$[(X_1 - k^*_1)_+, \ldots, (X_n - k^*_n)_+, | S > \text{VaR}_q(S)]$$

is comonotonic. According to Proposition 1 of Cheung (2009), it holds that

$$\left[ \sum_{j=1}^{n} (X_j - k^*_j)_+, | S > \text{VaR}_q(S) \right]$$

$$\approx \left[ (S - K)_+, | S > \text{VaR}_q(S) \right],$$

where $\approx_{s}$ represents both sides are almost surely equal. Therefore, we have

$$\text{Cov} \left[ \sum_{j=1}^{n} (X_j - k^*_j)_+, I(X_j \geq k^*_j) | S > \text{VaR}_q(S) \right]$$

$$= \text{Cov} \left[ (S - K)_+, I(S \geq k^*_j) | S > \text{VaR}_q(S) \right]$$

$$= \text{Cov} \left[ (S - K)_+, I(U \geq F_{X_n}(K)) | S > \text{VaR}_q(S) \right].$$


It is seen that Eq. (2.1) is fulfilled if \( k^* \) is a solution. We conclude that \( k^* \) is an optimal solution for Model (1.2). Further, the solution \( k^* \) is unique, as it does not depend on the parameter \( \beta \). Hence, the required result follows.

Theorem 3.4 presents a closed-form solution of capital allocations for the comonotonic risks. It might be a little surprising to observe that the optimal capital allocation rule based on Model (1.2) for comonotonic risks adopts the same formulas as that in Corollary 2.2, i.e., without the penalty on the tail variance. A careful checking of Theorem 2.1 reveals that, although the formulas are the same, the meanings are quite different for both scenarios. Corollary 2.2 presents the optimal capital allocations for any dependence structure by considering only the magnitude of tail risks. However, Theorem 3.4 presents the optimal capital allocations for the comonotonic risks by considering both the magnitude and variability of tail risks. But, for this particular dependence structure, the magnitude and variability of loss functions are minimized simultaneously, which explains the same optimal capital allocation formulas as in Corollary 2.2. One may wonder whether the magnitude and variability of loss functions could be minimized simultaneously for other general multivariate risks, i.e., \( \beta \) is irrelevant to the optimal capital allocations. The answer is negative from the examples in Section 4. In fact, the penalty parameter \( \beta \) has nonnegligible influence on the capital allocations.

4. Examples and applications

In this section, we present some examples of optimal capital allocations based on Model (1.2) for specific multivariate distributions. We will also apply the new capital allocation rule to real data from one insurance company.

4.1. Elliptical distributions

In the literature of insurance and actuarial science, the elliptical distribution has attracted much attention, mainly due to its mathematical tractability. It includes many well-known distributions, such as multivariate normal distribution, multivariate t distribution, multivariate logistic distribution, and multivariate exponential power distribution, etc. For more discussion of elliptical distribution, one may refer to Fang, Kotz, and Ng (1987) and Landsman and Valdez (2003).

In the following, we first give a brief review of some properties of elliptical distribution, which is pertinent to the discussion of our main results.

**Definition 4.1.** The random vector \( \mathbf{X} \) has a multivariate elliptical distribution, denoted by \( \mathbf{X} \sim \mathcal{E}_n(\mathbf{m}, \Sigma, \psi) \), if its characteristic function can be expressed as

\[
\phi_{\mathbf{X}}(\mathbf{t}) = \exp(i\mathbf{t}^T\mathbf{m}) \psi(\Sigma^{-1/2} \mathbf{t}/2)
\]

for some column-vector \( \mathbf{m} \), \( n \times n \) positive definite matrix \( \Sigma \), and characteristic generator \( \psi(\cdot) \).

It should be pointed out that not every multivariate elliptical distribution has a density function. If \( \mathbf{X} \sim \mathcal{E}_n(\mathbf{m}, \Sigma, \psi) \), and \( \mathbf{X} \) has a density \( f_X(x) \), then,

\[
f_X(x) = \frac{c_n}{\Sigma^{1/2}} g_n \left( \frac{1}{2}(x - \mu)^T \Sigma^{-1/2} \Sigma^{-1/2} (x - \mu) \right),
\]

where

\[
c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left( \int_0^\infty x^{n/2-1} g_n(x) \, dx \right)^{-1},
\]

and

\[
\int_0^\infty x^{n/2-1} g_n(x) \, dx < \infty,
\]

which guarantees \( g_n(x) \) to be the density generator. For this case, one may write \( \mathbf{X} \sim \mathcal{E}_n(\mu, \Sigma, g_n) \).

If the mean exists, we have \( \mathbb{E}(\mathbf{X}) = \mu \). The condition guarantees the existence of the covariance matrix \( |\psi'(0)| < \infty \) and hence

\[
\text{Cov}(\mathbf{X}) = -\psi'(0) \Sigma.
\]

Without loss of generality, in the following discussion, it is assumed that \( -\psi'(0) = 1 \), and hence \( \text{Cov}(\mathbf{X}) = \Sigma \). For the comprehensive discussion of properties of
elliptical distributions, please refer to Fang, Kotz, and Ng (1987).

We first recall the well-known property of elliptical distributions.

**Proposition 4.2.** If \( X \sim E_n(\mu, \Sigma, g_\mu) \), and \( A \) is some \( m \times n \) matrix of rank \( m \leq n \), and \( b \) some \( m \)-dimensional column-vector, then

\[
AX + b - E_n(A\mu + b, A\Sigma A^T, g_m).
\]

Next, we compute the key quantities, including \( \tilde{F}_{i,j}(\cdot) \), \( \text{ECT}_{i,j}(\cdot) \), and \( \text{Cov}_{+,+}(\cdot, \cdot) \) for the family of elliptical distributions, which would facilitate the computations of Eq. (2.1). Note that if \( X \sim E_n(\mu, \Sigma, g_\mu) \), then by Proposition 4.2, it holds that

\[
S - E_i(\mu_3, \Sigma_{3,5}, g_i),
\]

where \( \mu_3 = \Sigma_{1,1}^n \mu_1 \) and \( \sigma_{3,5} = \Sigma_{1,1}^n \sigma_1 \) with \( \sigma_5 = \text{Cov}(X_1, X_2) \). Further, by Xu and Mao (2013), we have

\[
(X_1, X_2 | S = s) - E_2(\mu_{1,2}, \Sigma_{1,2}, g_2),
\]

where

\[
\mu_{1,2} = \left( \begin{array}{c} \mu_{1,2}^1 \\ \mu_{1,2}^2 \end{array} \right) = \left( \begin{array}{c} s(\sigma_{1,2}/\sigma_{3,5}) + \mu_1 - \mu_{3,5} \sigma_{1,2}/\sigma_{3,5} \\ s(\sigma_{1,2}/\sigma_{3,5}) + \mu_2 - \mu_{3,5} \sigma_{1,2}/\sigma_{3,5} \end{array} \right),
\]

and

\[
\Sigma_{1,2} = \left( \begin{array}{cc} \sigma_{1,2} & \sigma_{1,2}^2/\sigma_{3,5} \\ \sigma_{1,2} & \sigma_{1,2}^2/\sigma_{3,5} \end{array} \right) = \left( \begin{array}{cc} \sigma_1 - \sigma_{1,2} \sigma_{1,2}/\sigma_{3,5} & \sigma_2 - \sigma_{1,2} \sigma_{1,2}/\sigma_{3,5} \\ \sigma_2 - \sigma_{1,2} \sigma_{1,2}/\sigma_{3,5} & \sigma_3 - \sigma_{1,2} \sigma_{1,2}/\sigma_{3,5} \end{array} \right).
\]

The survival function of \( X_1 | S > \text{VaR}_q(S) \) can be computed as

\[
\tilde{F}_{i,j}(k_i) = \int_k^\infty f_i(x| S > \text{VaR}_q(S)) dx = \int_k^\infty \int_{\text{VaR}_q(S)} f_{i,j}(x| s) dF_j(s| S > \text{VaR}_q(S)) dx
\]

where

\[
\text{VaR}_q(S) = \frac{c_1}{\sqrt{\sigma_{3,5}^2 \sigma_{3,5}^2}} g_i \left( \frac{(s - \mu_{3,5})^2}{2 \sigma_{3,5}^2} \right)
\]

and

\[
0 < q < 1
\]

\[
\text{ECT}_{i,j}(k_i) = \int_k^\infty f_i(x| S > \text{VaR}_q(S)) dx
\]

where \( q = (k_i - \mu_{3,5})/\sqrt{\sigma_{3,5}^2} \) with \( w = \sqrt{\sigma_{3,5}^2} + \mu_{3,5} \), and \( w' = (\text{VaR}_q(S) - \mu_{3,5})/\sqrt{\sigma_{3,5}^2} \). The notation \( f_i(x| S > \text{VaR}_q(S)) \) represents the density function of \( [X_1 | S > \text{VaR}_q(S)] \), and \( F_j(s| S > \text{VaR}_q(S)) \) represents the distribution function of \( [S| S > \text{VaR}_q(S)] \).

Next, we provide a simple form for computing \( \text{ECT}_{i,j}(\cdot) \).

\[
\text{ECT}_{i,j}(k_i) = \mathbb{E}[X_i - k_i, | S > \text{VaR}_q(S)]
\]

where

\[
\mathbb{E}[X_i - k_i, | S > \text{VaR}_q(S)]
\]

The conditional covariance \( \text{Cov}_{+,+}(k_i, k_j) \) can be represented as

\[
\text{Cov}_{+,+}(k_i, k_j) = \mathbb{E}[X_i - k_i, 1(X_i \geq k_j) | S > \text{VaR}_q(S)] - \text{ECT}_{i,j}(k_i, k_j).
\]

Note that

\[
\mathbb{E}[X_i - k_i, 1(X_i \geq k_j) | S > \text{VaR}_q(S)]
\]

where

\[
\text{ECT}_{i,j}(k_i, k_j)
\]

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\]

where

\[
\text{ECT}_{i,j}(k_i, k_j)
\]
where \( f_{ij}(\cdot, \cdot | S > \text{VaR}_q(S)) \) is the joint density function of \([(X_i, X_j) | S > \text{VaR}_q(S)]\), which has the following form

\[
f_{ij}(x_i, x_j | S > \text{VaR}_q(S)) = \int_{\text{VaR}_q(S)} f_{ij}(x_i, x_j | S = s) dF_S(s | S > \text{VaR}_q(S)) = \frac{1}{1-q} \frac{c_i}{\sqrt{\sigma_{ij}}} \int_{\text{VaR}_q(S)} f_{ij}(x_i, x_j | S = s) g_i \left( \frac{(s - \mu_i)^2}{2\sigma_{ij}^2} \right) ds,
\]

where \( f_{ij}(x_i, x_j | S = s) \) is the density function of \([(X_i, X_j) | S = s]\), which is the bivariate elliptical distribution by Eq. (4.2). One may easily implement the forms of Eqs. (4.3), (4.4) and (4.5) into Eq. (2.1) to derive the solutions.

In the following, we present a numerical example to study the optimal capital allocations based on Model (1.2).

**Example 4.3.** An \( n \)-dimensional multivariate student-\( t \) distribution belongs to an elliptical family if its density generator can be expressed as

\[
g_n(x) = \left(1 + \frac{x}{k_p}\right)^{-p}
\]

where \( p > n/2 \), and \( k_p \) is some constant depending on \( p \). For simplicity, we assume that \( p = n + v \) with the degree of freedom \( v \), and \( k_p = v/2 \). The joint density has the following form:

\[
f(x) = \frac{c_n}{\sqrt{\Sigma}} \left[1 + \frac{(x - \mu)^T \Sigma^{-1} (x - \mu)}{v}\right]^{-(n+v)/2}, \tag{4.6}
\]

where

\[
c_n = \frac{\Gamma((n + v)/2)}{\Gamma(v/2)} (\pi v)^{-n/2}.
\]

Next, by using a specific example, we discuss how the different factors affect the optimal capital allocations based on Model (1.2), which include the dependence, variance penalty parameter \( \beta \), risk level \( q \), and heavy tail. Assume that an insurance company has three business lines \((X_1, X_2, X_3)\), which follow the multivariate student \( t \) distribution with mean vector

\[
\mu = (6, 10, 5),
\]

and

\[
\Sigma = \begin{bmatrix} 1 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & 3 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 1 \end{bmatrix}.
\]

The total capital is assumed to be \( K = 25 \). In the following, we examine several scenarios by varying the parameters \( \sigma_{12}, \sigma_{13}, \sigma_{23} \). The results are summarized in Table 1, which are thoroughly discussed as follows.

- **Dependence effect.** To study the dependence effect, we vary the values of \( \sigma_{12}, \sigma_{13}, \sigma_{23} \). As seen from Table 1, when \( \sigma_{12} \) ranges from \{0, .5, 1.5\} and \( \sigma_{23} \) from \{-5, 0, .5\}, the more dependence results the more capital requirement. For example, for the case \((v, \beta, q) = (5, .01, .95)\), when \( \sigma_{12} \) changes from 0 to .5, and \( \sigma_{23} \) changes from 0 to -.5, it is found that the required capital for risk \( X_1 \) increases from 6.618 (26.47%) to 7.280 (29.12%), but risk \( X_2 \) reduces from 5.618 (22.47%) to 4.963 (19.85%); when \( \sigma_{13} \) changes from 0 to .1, which indicates increasing the dependence between \( X_1 \) and \( X_3 \), it is found out that the capital requirement for \( X_3 \) increases from 4.793 (19.17%) to 4.849 (19.40%).

- **Penalty \( \beta \).** From Table 1, it is observed that when \( \beta \) changes from .01 to .1, the capital requirement on \( X_1 \) increases for all the cases. This is very reasonable since \( X_1 \) is the riskiest one. For example, for \((\sigma_{12}, \sigma_{13}, \sigma_{23}) = (1.5, .1, .5)\) and \((v, q) = (5, 99)\), when \( \beta \) changes from .01 to .1, the allocation amount changes from 14.074 (56.30%) to 14.216 (56.86%), which reflects the penalty on the variance of new model as expected.

- **Risk level \( q \).** Table 1 presents the capital allocations for two risk levels \( q = .95 \) and \( q = .99 \). The
risk level increases, reflecting that the insurance company is more conservative about the risk. Hence the insurance company may be willing to allocate the more capital to the business lines with larger risks. It is seen from Table 1 that the capital allocation to \( X_2 \) increases for all cases, which meets the aim of controlling the risk. For example, it is seen that when \((\sigma_{12}, \sigma_{13}, \sigma_{23}) = (1.5, .1, .5)\), the capital requirement of \( X_2 \) is 14.216 (56.86%) based on \( q = .99 \) compared to that of 13.013 (52.05%) based on \( q = .95 \).

- Tail effect. The cases of \( v = 5 \) and \( v = 50 \) are used to calculate the capital allocations in Table 1, which represent different tail thickness of marginal distributions. It is known that when \( v \) is smaller, the tail probability of t distribution is larger. It is clearly seen from Table 1 that when \( v \) is smaller, the capital allocation requirement is larger. For example, it is seen that when \((\sigma_{12}, \sigma_{13}, \sigma_{23}) = (1.5, .1, .5)\), the capital requirement of \( X_2 \) is 12.625 (50.40%) based on \((v, \beta, q) = (5, .01, .95)\) compared to that of 12.435 (49.74%) based on \((v, \beta, q) = (50, .01, .95)\).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( k_1^* )</th>
<th>( k_2^* )</th>
<th>( k_3^* )</th>
<th>( k_4^* )</th>
<th>( k_5^* )</th>
<th>( k_6^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 0))</td>
<td>6.618 12.763 5.618 6.296 13.408 5.296</td>
<td>26.47% 51.05% 22.47% 25.18% 53.63% 21.18%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((1.5, 0, .5))</td>
<td>7.109 13.099 4.793 7.059 14.377 3.564</td>
<td>28.44% 52.40% 19.17% 28.24% 57.51% 14.26%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((1.5, .1, .5))</td>
<td>7.195 12.956 4.849 7.251 14.074 3.674</td>
<td>28.78% 51.83% 19.40% 29.00% 56.30% 14.70%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((5, 0, -.5))</td>
<td>7.280 12.758 4.963 7.393 13.379 4.229</td>
<td>29.12% 51.03% 19.85% 29.57% 53.52% 16.92%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((5, .1, -.5))</td>
<td>7.339 12.625 5.036 7.505 13.154 4.341</td>
<td>29.36% 50.40% 20.14% 30.02% 52.62% 17.36%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Optimal capital allocations (amounts and percentages) based on the TMV model (1.2) with a total capital \( K = 25 \).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( v = 5, \beta = .01, q = .95 )</th>
<th>( v = 5, \beta = .01, q = .99 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 0))</td>
<td>6.599 12.801 5.599 6.275 13.451 5.275</td>
<td>26.40% 51.20% 22.40% 25.10% 53.80% 21.10%</td>
</tr>
<tr>
<td>((1.5, 0, .5))</td>
<td>7.122 13.161 4.717 7.086 14.518 3.936</td>
<td>28.49% 52.64% 18.87% 28.34% 58.07% 13.58%</td>
</tr>
<tr>
<td>((1.5, .1, .5))</td>
<td>7.213 13.013 4.773 7.285 14.216 3.499</td>
<td>28.85% 52.05% 19.09% 29.14% 56.86% 14.00%</td>
</tr>
<tr>
<td>((5, 0, -.5))</td>
<td>7.306 12.797 4.897 7.431 13.426 4.143</td>
<td>29.22% 51.19% 19.59% 29.72% 53.70% 16.57%</td>
</tr>
<tr>
<td>((5, .1, -.5))</td>
<td>7.369 12.656 4.975 7.548 13.191 4.261</td>
<td>29.48% 50.62% 19.90% 30.19% 52.76% 17.04%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( v = 50, \beta = .01, q = .95 )</th>
<th>( v = 50, \beta = .01, q = .99 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 0))</td>
<td>6.731 12.539 5.731 6.575 12.851 5.575</td>
<td>26.92% 50.16% 22.92% 26.30% 51.40% 22.30%</td>
</tr>
<tr>
<td>((1.5, 0, .5))</td>
<td>7.109 12.752 5.138 7.078 13.350 4.572</td>
<td>28.44% 51.01% 20.55% 28.31% 53.40% 18.29%</td>
</tr>
<tr>
<td>((1.5, .1, .5))</td>
<td>7.170 12.651 5.180 7.189 13.170 4.641</td>
<td>28.68% 50.60% 20.72% 28.76% 52.68% 18.56%</td>
</tr>
<tr>
<td>((5, 0, -.5))</td>
<td>7.228 12.535 5.237 7.281 12.833 4.885</td>
<td>28.91% 50.14% 20.95% 29.12% 51.33% 19.54%</td>
</tr>
<tr>
<td>((5, .1, -.5))</td>
<td>7.274 12.435 5.291 7.352 12.689 4.959</td>
<td>29.10% 49.74% 21.16% 29.41% 50.76% 19.84%</td>
</tr>
</tbody>
</table>
From this example, it is observed that Model (1.2) has many intriguing properties, such as reflecting the effects of dependence, penalty, tail, and risk level. The numerical results also possess the intuitive explanations. It should be pointed out that the elliptical distributions discussed here are symmetric. In the following section, we discuss a family of skewed multivariate distributions.

### 4.2. Multivariate skew-t family

Insurance risks may have skewed distributions, for which the symmetric distributions such as multivariate normal or t distributions are not appropriate models for insurance risks or losses. Therefore, in the literature, the multivariate skewed distributions have been proposed as alternatives to model such risks. Among many multivariate skewed distributions, the multivariate skew-t distribution has been favored since it provides the benefit of flexibility with regard to skewness and thickness of the tails. It allows unlimited range for the indices of skewness and kurtosis for the individual components. For a comprehensive discussion about skewed-distribution family, one may refer to Azzalini (2014).

In the following, we give the definition of a multivariate skew-t distribution.

**Definition 4.4.** The random vector \( \mathbf{X} \) has a multivariate skew-t distribution, denoted by \( \mathbf{X} \sim ST(\xi, \Omega, \alpha, \nu) \), if its density function can be expressed as

\[
f_X(x) = 2 t_n(x; \xi, \Omega, \nu) T_1 \left( \alpha^T \Omega^{-1} (x - \xi) \left( \frac{v + p}{v + Q(x)} \right)^{\nu/2}; \nu + n \right)
\]

where \( Q(x) = (x - \xi)^T \Omega^{-1} (x - \xi) \), \( \alpha \in \mathbb{R}^n \) is the shape parameter, and

\[
t_n(x; \xi, \Omega, \nu) = \frac{\Gamma((\nu + n)/2)}{|\Omega|^{\nu/2} (\nu \pi)^{n/2} \Gamma(\nu/2)} \left( 1 + \frac{Q(x)}{\nu} \right)^{-(\nu+n)/2}
\]

represents the density function of usual \( n \)-dimensional Student’s \( t \) distribution with location \( \xi \), positive definite \( n \times n \) dispersion matrix \( \Omega \), and \( T_1(\cdot; \nu) \) denotes the univariate standard Student’s \( t \) cumulative distribution function with degrees of freedom \( \nu > 0 \).

It should be mentioned that although the multivariate skew-t distribution has many similar properties to the multivariate t distribution, it does not have the preservation property that the conditional distribution is still in the original family of distributions. Therefore, the analytical forms of the key quantities in Eq. (2.1) are infeasible to derive. Instead, we propose to use the Monte Carlo simulation method to compute the key quantities. Specifically, we generate 1,000,000 observations from the multivariate skew-t distribution to compute \( F_{-i,S}(.), ECT_{S}(.), \text{ and } \text{Cov}^{+}_{S}(., .) \), which are illustrated by the following specific example.

**Example 4.5.** Assume that an insurance company has three business lines \( (X_1, X_2, X_3) \), which follow the multivariate skew-t distribution with location parameters

\[
\xi = (6, 10, 5),
\]

and shape parameters

\[
\alpha = (10, 30, 20).
\]

The dispersion matrix is assumed to be

\[
\Omega = \begin{pmatrix}
1 & \omega_{12} & \omega_{13} \\
\omega_{21} & 3 & \omega_{23} \\
\omega_{31} & \omega_{32} & 1
\end{pmatrix}.
\]

We note that although the dispersion matrix is not the covariance matrix, it is linearly related to the covariance matrix, which still reflects the dependence between \( (X_1, X_2, X_3) \). The specific relation may be found in Eq. (6.26) of Azzalini (2014).

We use the same parameters as that in Table 1 to compute the optimal capital allocations based on Eq. (2.1). The results are summarized in Table 2.
4.3. Comparisons to other methods

In this section, we compare the TMV model (1.2) to several models frequently used in the literature. For comprehensive reviews on the methodologies of capital allocations, one may refer to Dhaene et al. (2012), and Bauer and Zanjani (2013). Specifically, the capital allocation rules considered in this section include:

(a) Haircut allocation:

\[ k_i = \frac{F_{X_i}^{-1}(q)}{\sum_{j=1}^{n} F_{X_j}^{-1}(q)} K \]

For the multivariate skew-t distributions, we may draw similar conclusions to those in Example 4.3, i.e., the dependence, penalty parameter \( \beta \), risk level, and tail thickness all have significant effects on the capital allocations based on Model (1.2). It is interesting to observe that the capital requirements on risk \( X_2 \) in Table 2 are larger than the corresponding ones in Table 1. This may be intuitively explained by the large skewness of risk \( X_2 \). Hence, in practice, one should always seek a suitably skewed distribution if the faced risks are skewed.

Table 2. Optimal capital allocations (amounts and percentages) based on the TMV model (1.2) with a total capital \( K = 25 \).

<table>
<thead>
<tr>
<th>( (\omega_{i1}, \omega_{i2}, \omega_{i3}) )</th>
<th>( k_1^* )</th>
<th>( k_2^* )</th>
<th>( k_3^* )</th>
<th>( k_1^* )</th>
<th>( k_2^* )</th>
<th>( k_3^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>*( (0, 0, 0) )</td>
<td>6.448</td>
<td>13.094</td>
<td>5.458</td>
<td>6.032</td>
<td>13.902</td>
<td>5.066</td>
</tr>
<tr>
<td>25.79%</td>
<td>52.38%</td>
<td>21.83%</td>
<td>24.13%</td>
<td>55.61%</td>
<td>20.26%</td>
<td></td>
</tr>
<tr>
<td>*( (1.5, 0, .5) )</td>
<td>7.120</td>
<td>13.769</td>
<td>4.111</td>
<td>7.194</td>
<td>15.226</td>
<td>2.580</td>
</tr>
<tr>
<td>28.48%</td>
<td>55.08%</td>
<td>16.44%</td>
<td>28.78%</td>
<td>60.90%</td>
<td>10.32%</td>
<td></td>
</tr>
<tr>
<td>*( (1.5, 1, .5) )</td>
<td>7.258</td>
<td>13.547</td>
<td>4.194</td>
<td>7.432</td>
<td>14.788</td>
<td>2.780</td>
</tr>
<tr>
<td>29.03%</td>
<td>54.19%</td>
<td>16.78%</td>
<td>29.73%</td>
<td>59.15%</td>
<td>11.12%</td>
<td></td>
</tr>
<tr>
<td>*( (5, 0, -.5) )</td>
<td>7.349</td>
<td>13.073</td>
<td>4.578</td>
<td>7.499</td>
<td>13.828</td>
<td>3.673</td>
</tr>
<tr>
<td>29.40%</td>
<td>52.30%</td>
<td>18.31%</td>
<td>30.00%</td>
<td>55.31%</td>
<td>14.69%</td>
<td></td>
</tr>
<tr>
<td>*( (5, 1, -.5) )</td>
<td>7.437</td>
<td>12.889</td>
<td>4.674</td>
<td>7.645</td>
<td>13.529</td>
<td>3.826</td>
</tr>
<tr>
<td>29.75%</td>
<td>51.56%</td>
<td>18.70%</td>
<td>30.58%</td>
<td>54.12%</td>
<td>15.30%</td>
<td></td>
</tr>
</tbody>
</table>

Parameters: \( \nu = 5, \ \beta = .1, \ q = .95 \)
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VOLUME 10/ISSUE 2

The total capital is also assumed to be $K = 25$, and the parameter $\beta = .01$. We calculate the optimal capitals based on different allocation rules. The results are presented in Table 3. It is seen that the quantile allocation rule allocates the smallest amount of capital to risk $X_2$ (42.43%) compared to the other allocation rules. In particular, the allocation amount based on the quantile rule does not change when the risk level $q$ changes from .95 to .99. The covariance allocation rule allocates the largest amount of capital to risk $X_2$ (57.91%) compared to the other allocation rules, but it cannot reflect the risk level. Compared to CTE and TMV, the haircut rule allocates a relatively smaller amount of capital to $X_2$. It is interesting to observe that when the risk level increases from .95 to .99, the allocation amount for the riskiest $X_2$ decreases from 46.60% to 46.50%. Therefore, the haircut allocation rule does not reflect the risk level very well. The CTE and TMV are similar from the perspectives of allocation amounts and risk levels. Both of them allocate relatively larger capitals to risk $X_2$, and the allocation amounts increase when the risk level increases from .95 to .99. However, the capital based on TMV model increases from 50.40% to 52.62%, while the

where $F_X^{-1}(q)$ is the left continuous inverse of the distribution function of $X_i$ at $q > 0$;

(b) Quantile allocation:

$$k_i = \frac{F_X^{-1}(F_i(K))}{\sum_{j=1}^{n} F_X^{-1}(F_j(K))} K$$

where $F_i(K) = \mathbb{P}(\sum_{i=1}^{n} F_X^{-1}(U) \leq K)$, and $U$ is a uniform random variable on $(0, 1)$;

(c) Covariance allocation:

$$k_i = \frac{\text{Cov}(X_i, S)}{\sum_{j=1}^{n} \text{Cov}(X_j, S)} K$$

where $S = \sum_{i=1}^{n} X_i$;

(d) CTE (conditional tail expectation) allocation

$$k_i = \frac{\mathbb{E}[X_i | S > F_S^{-1}(q)]}{\sum_{j=1}^{n} \mathbb{P}[X_j | S > F_S^{-1}(q)]} K$$

where $F_S^{-1}(q)$ is the left continuous inverse of the distribution function of $S$ at $q > 0$.

**Example 4.6.** For the purpose of comparison, we use the same distribution as Example 4.3. That is, three business lines ($X_1, X_2, X_3$) follow a multivariate Student $t$ distribution with mean vector

$$\mu = (6, 10, 5),$$

and

$$\Sigma = \begin{pmatrix} 1 & .5 & .1 \\ .5 & 3 & -.5 \\ .1 & -.5 & 1 \end{pmatrix}.$$
By assuming that the joint distribution of these ten random variables follows a multivariate normal distribution, Panjer (2002) discusses the optimal allocation problems for this data set. We assume that the joint distribution follows a multivariate Student-t distribution with density function defined in Eq. (4.6). Since the original data is not available to us, we use $n = 9$ and $n = 50$ for the data set, which represent small and large degree of freedoms, separately.

The total capital $K$ is assumed to be 147 million, which is around one standard deviation of estimated means larger than the total sum of estimated means 134.13 million. This value is slightly larger than the VaR.95 ($v = 9$ based on $n = 9$), where $S = X_1 + \ldots + X_{10}$. Table 4 summarizes the optimal capital allocations for various scenarios based on $q = .99$.

From Table 4, it is observed that overall the larger risks are allocated with more capitals. It is seen from the covariance matrix that $X_2$ has the largest mean and variance, and it has a positive correlation with relatively large risks, say larger than 9 million ($X_4$, $X_6$, $X_7$), but it is uncorrelated with $X_1$, and is negatively correlated with $X_{10}$. When $\beta$ is increasing, the capital requirement on $X_2$ is increasing for both of $v = 50$ and $v = 9$; the capitals for $X_2$ with $v = 9$ are larger than the corresponding ones with $v = 50$. This observation reflects that the model penalizes the large variance and heavy tail. For risks $X_1$ and $X_6$ with estimated means 25.69 and 24.05 millions, the capital requirements are 26.711 (18.17%) and 27.116 (18.45%) millions for CTE only increases from 50.60% to 51.58%. This reflects the advantage of TMV model in quickly responding to a large risk level. It is also seen that the allocated capital for risk $X_3$ based on the TMV model decreases by 2.78% while the allocated capital based on the CTE decreases by 1.19% for $X_3$. This is because the new TMV model takes into account the negative dependence between $X_2$ and $X_3$ for allocations. To conclude, compared to the other models, the new Model (1.2) has many desired properties, such as reflecting the effects of dependence, and risk level.

4.4. Real data analysis

In this section, we analyze a real insurance data set presented in Panjer (2002). The total number of business lines is 10 with

$$ X^T = (X_1, \ldots, X_{10}), $$

which represent a range of insurance and other related financial products. The estimated mean vector (million) is

$$ \mu = (25.69, 37.84, 0.85, 12.70, 0.15, 24.05, 14.41, 4.49, 4.39, 9.56). $$

The correlation matrix was reported in Panjer (2002) and Valdez and Chernih (2003) reported the covariance matrix, which is reproduced here for the sake of convenience.

$$
\begin{bmatrix}
7.24 & 0 & 0.07 & -0.07 & 0.28 & -2.71 & -0.51 & 0.28 & 0.23 & -0.21 \\
20.16 & 0.05 & 1.6 & 0.05 & 1.39 & 1.14 & -0.91 & -0.81 & -1.74 \\
0.04 & 0 & -0.01 & 0.08 & 0.01 & -0.02 & -0.02 & -0.07 \\
1.74 & 0.17 & 0.26 & 0.19 & -0.14 & 0.18 & -0.79 \\
0.32 & -0.24 & 0.01 & -0.02 & 0.08 & -0.01 \\
14.98 & 0.43 & -0.33 & -1.89 & -1.6 \\
2.53 & -0.38 & 0.13 & 0.58 \\
0.92 & -0.16 & -0.4 & 1.12 & 0.58 \\
6.71 & & & & & & & & & 
\end{bmatrix}
$$
the works by Laeven and Goovaerts (2004), Dhaene et al. (2012), and Xu and Mao (2013), which capture both magnitude and variability of tail risks. As seen from the numerical evidence, the TMV model has many intriguing properties, such as penalizing the large risk, variance, positive dependence, and reflecting the tail risk level. It also provides many intuitive explanations on the optimal capital allocations. The penalization parameter $\beta$, which is either determined by the historical data or by the experience of the decision maker, provides an additional flexibility for controlling the tail variability. Since the analytical solutions for the TMV model is infeasible, we explore the general equations which could be easily implemented in the software (R code is available upon request). It may be interesting to comprehensively compare the TMV model to those in the literature (Bauer and Zanjani 2013) and use the TMV model for DHS capital allocation. The preliminary study shows that the TMV model provides some promising results, which is currently being pursued, and will be reported when it is completed.

5. Conclusion

In this paper, we have suggested a new capital allocation rule which stems from the tail mean-variance premium calculation principle. It is also a variation of

Table 4. Optimal capital allocations (amounts and percentages) for various parameters based on TMV model (1.2) with a total capital $K = 147$ and $q = .99$.

<table>
<thead>
<tr>
<th>$\nu = 50$</th>
<th>$\beta = .01$</th>
<th>$\beta = .1$</th>
<th>$\beta = .5$</th>
<th>$\nu = 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>18.18%</td>
<td>18.17%</td>
<td>18.18%</td>
<td>18.10%</td>
</tr>
<tr>
<td>$k_2$</td>
<td>45.358</td>
<td>45.476</td>
<td>45.655</td>
<td>45.732</td>
</tr>
<tr>
<td></td>
<td>30.86%</td>
<td>30.94%</td>
<td>31.06%</td>
<td>31.11%</td>
</tr>
<tr>
<td>$k_3$</td>
<td>840</td>
<td>840</td>
<td>843</td>
<td>832</td>
</tr>
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$v = 50$, respectively. The capital requirement on risk $X_6$ slightly decreases when $\beta$ changes from .01 to .5, which may be caused by the correlations with the other risks. For the case of $v = 9$, it is observed that the capital requirements for risks $X_1$ and $X_6$ both increase when $\beta$ changes from .01 to .5, which may reflect the penalty on the variability again. It is interesting to observe that when $\beta$ changes from .01 to .1 with $v = 9$, the capital requirement on $X_1$ is slightly less, while on $X_6$ it is slightly more. It may be explained by noting that the variance of $X_1$ is less than that of $X_6$, and further $X_6$ is positively correlated with $X_2$. For risk $X_{10}$, it is seen that it is negatively correlated with $(X_1, X_2, X_4, X_6)$, and therefore, it is not surprising to observe that the capital requirements all decrease when $\beta$ increases.
6. Acknowledgment

The author thanks the editor and two anonymous reviewers for their constructive comments that helped to improve the presentation of this paper. In particular, Section 4.3 was added based on the suggestion from a referee. This work was partly supported by the Casualty Actuarial Society through the Individual Grants Competition.

Appendix

Proof of Theorem 2.1: Define

\[ f(k) = E\left[\sum_{i=1}^{n}(X_i - k_i)_+ | S > \text{VaR}_q(S)\right] \]
\[ + \beta \text{Var}\left[\sum_{i=1}^{n}(X_i - k_i)_+ | S > \text{VaR}_q(S)\right]. \]

and

\[ h(k) = K - \sum_{i=1}^{n} k_i. \]

Let

\[ L(k, \lambda) = f(k) + \lambda h(k). \]

According to Kuhn-Tucker theory (Bertsekas 1999), we need to solve the following equations, for \( l = 1, \ldots, n, \)

\[ \frac{\partial L(k, \lambda)}{\partial k_i} = 0, \quad \frac{\partial L(k, \lambda)}{\partial \lambda} = 0. \]

We first observe that

\[ \frac{\partial \text{ECT}_S(k_i)}{\partial k_i} = -\bar{F}_{1,S}(k_i), \]

and

\[ \text{Var}\left[(X_i - k_i)_+ | S > \text{VaR}_q(S)\right] \]
\[ = E\left[(X_i - k_i)_+^2 | S > \text{VaR}_q(S)\right] - \left[\text{ECT}_S(k_i)\right]^2. \]

Therefore, it holds that

\[ \frac{\partial \text{Var}\left[(X_i - k_i)_+ | S > \text{VaR}_q(S)\right]}{\partial k_i} \]
\[ = -2\bar{F}_{1,S}(k_i) \text{ECT}_S(k_i). \]

Further, for any \( j = 2, \ldots, n, \) we have

\[ \frac{\partial \text{Cov}\left\{(X_j - k_j)_+,(X_i - k_i)_+ | S > \text{VaR}_q(S)\right\}}{\partial k_i} \]
\[ = -\int_{k_i}^{\infty} \int_{k_i}^{\infty} f_{i,j}(x_i, x_j) \text{I}(x_i \geq k_i) | S > \text{VaR}_q(S) | \, dx_i \, dx_j \]
\[ + \bar{F}_{1,S}(k_i) \text{ECT}_S(k_i) \]
\[ = -\text{Cov}\left\{(X_j - k_j)_+, I(X_i \geq k_i) | S > \text{VaR}_q(S)\right\} \]
\[ - \bar{F}_{1,S}(k_i) \text{ECT}_S(k_i) + \bar{F}_{1,S}(k_i) \text{ECT}_S(k_j) \]
\[ = -\text{Cov}_{+,+}(k_j, k_i), \]

where \( f_{i,j}(\cdot, \cdot | S > \text{VaR}_q(S)) \) is the joint density of \([X_i, X_j] | S > \text{VaR}_q(S)\]. Therefore, we have

\[ \frac{\partial f(k)}{\partial k_i} = -\bar{F}_{1,S}(k_i) - 2\beta \bar{F}_{1,S}(k_i) \text{ECT}_S(k_i) \]
\[ - 2\beta \sum_{j=2}^{n} \text{Cov}_{+,+}(k_j, k_i) \]
\[ = -\bar{F}_{1,S}(k_i) - 2\beta \text{ECT}_S(k_i) \]
\[ + 2\beta \bar{F}_{1,S}(k_i) \text{ECT}_S(k_i) \]
\[ - 2\beta \sum_{j=2}^{n} \text{Cov}_{+,+}(k_j, k_i) \]
\[ = -\bar{F}_{1,S}(k_i) - 2\beta \text{Cov}\left\{(X_i - k_i)_+, I(X_i \geq k_i) | S > \text{VaR}_q(S)\right\} \]
\[ - 2\beta \sum_{j=2}^{n} \text{Cov}_{+,+}(k_j, k_i) \]
\[ = -\bar{F}_{1,S}(k_i) - 2\beta \sum_{j=1}^{n} \text{Cov}_{+,+}(k_j, k_i). \]

For \( l = 1, 2, \ldots, n, \) it follows that

\[ \frac{\partial L(k, \lambda)}{\partial k_i} = -\bar{F}_{1,S}(k_i) - 2\beta \sum_{j=1}^{n} \text{Cov}_{+,+}(k_j, k_i) - \lambda. \]
Therefore, the optimal solutions should satisfy the following equations:

\[
\tilde{F}_{l,S}(k^*_i) + 2\beta \sum_{j=1}^{n} \text{Cov}_{i,S}(k^*_j, k^*_i) = \tilde{F}_{l,S}(k^*_i) + 2\beta \sum_{j=1}^{n} \text{Cov}_{i,S}(k^*_j, k^*_i),
\]

and

\[
k^*_1 + \ldots + k^*_n = K.
\]

Next, we discuss the uniqueness condition of solutions. For any \( l \neq j \), it holds that

\[
\frac{\partial^2 L(k, \lambda)}{\partial k_i \partial k_j} = -2\beta \frac{\partial \text{Cov}_{i,S}(k_i, k_j)}{\partial k_j}
\]

\[
= 2\beta \text{Cov}[I(X_i > k_j), I(X_i > k_i) \mid S > \text{VaR}_q(S)].
\]

For \( l = 1, \ldots, n \), the second derivative of \( L(k, \lambda) \) is

\[
\frac{\partial^2 L(k, \lambda)}{\partial^2 k_l} = f_{l,S}(k_l) - 2\beta \sum_{j=1}^{n} \frac{\partial \text{Cov}_{i,S}(k_j, k_l)}{\partial k_l},
\]

where \( f_{l,S}(\cdot) \) represents the density function of \( [X_l \mid S > \text{VaR}_q(S)] \). Note that

\[
\sum_{j=1}^{n} \frac{\partial \text{Cov}_{i,S}(k_j, k_l)}{\partial k_l} = f_{l,S}(k_l) \text{ECT}_S(k_l)
\]

\[
- F_{l,S}(k_l) \overline{F}_{l,S}(k_l) + \sum_{j=1}^{n} \frac{\partial \text{Cov}_{i,S}(k_j, k_l)}{\partial k_j},
\]

and

\[
\sum_{j=1}^{n} \frac{\partial \text{Cov}_{i,S}(k_j, k_l)}{\partial k_l} = -f_{l,S}(k_l) \sum_{j=1}^{n} \mathbb{E}[(X_j - k_j)_+] X_i = k_l, S > \text{VaR}_q(S) - \text{ECT}_S(k_l)\}.
\]

We have

\[
\frac{\partial^2 L(k, \lambda)}{\partial^2 k_l} = f_{l,S}(k_l) - 2\beta f_{l,S}(k_l) \sum_{j=1}^{n} \text{ECT}_S(k_j)
\]

\[
+ 2\beta f_{l,S}(k_l) \overline{F}_{l,S}(k_l) + 2\beta f_{l,S}(k_l)
\]

\[
\sum_{j=1}^{n} \mathbb{E}[(X_j - k_j)_+ X_i = k_l, S > \text{VaR}_q(S)]
\]

\[
= 2\beta \text{Var}\{I(X_i > k_j) \mid S > \text{VaR}_q(S)\}
\]

\[
+ f_{l,S}(k_l) - 2\beta f_{l,S}(k_l) \sum_{j=1}^{n} \text{ECT}_S(k_j)
\]

\[
+ 2\beta f_{l,S}(k_l) \sum_{j=1}^{n} \mathbb{E}[(X_j - k_j)_+ X_i = k_l, S > \text{VaR}_q(S)]
\]

\[
= 2\beta \text{Var}\{I(X_i > k_j) \mid S > \text{VaR}_q(S)\}
\]

\[
+ f_{l,S}(k_l) \Delta_l,
\]

where

\[
\Delta_l = 1 - 2\beta \sum_{j=1}^{n} \text{ECT}_S(k_j) + 2\beta \sum_{j=1}^{n} \mathbb{E}[(X_j - k_j)_+ X_i = k_l, S > \text{VaR}_q(S)].
\]

Therefore, the Hessian matrix of the optimal solutions can be represented as

\[
H = 2\beta \text{Cov}[(I(X_i \geq k^*_1), \ldots, I(X_n \geq k^*_n))] S > \text{VaR}_q(S)] + \text{diag}(\Delta^*_S),
\]

where \( \text{diag}(\Delta^*_S) \) means the diagonal matrix with diagonal elements \( \Delta^*_S, l = 1, \ldots, n \). Hence, if

\[
\Delta^*_S = 1 - 2\beta \sum_{j=1}^{n} \text{ECT}_S(k^*_j) + 2\beta \sum_{j=1}^{n} \mathbb{E}[(X_j - k^*_j)_+ X_i = k^*_l, S > \text{VaR}_q(S)] > 0,
\]

then \( H \) is a positive definite matrix, as the covariance matrix is positive semi-definite. Since the set

\[
\{k \mid k_1 + k_2 + \ldots + k_n = K\}
\]

is convex, the optimal solution in Eq. (2.1) should also be a globe optimal solution.

The required result follows immediately.

\[\blacksquare\]

**References**

