Reciprocal Reinsurance Treaties Under an Optimal and Fair Joint Survival Probability

by Kchouk Bilel and Mélina Mailhot

ABSTRACT

In this paper, we study reinsurance treaties between an insurer and a reinsurer, considering both parties' interests. Most papers only focus on the insurer's point of view. The latest research considering both sides has considerably oversimplified the joint survival function. This situation leads to an unrealistic optimal solution; one of the parties can make risk-free profits while the other bears all the risk. Here, we define and optimize a fair joint survival probability for a reciprocal reinsurance treaty, under the expected value principle, for both quota-share and stop-loss reinsurance contracts.

KEYWORDS

Reinsurance treaties, optimization, joint survival, expected value principle

1. Introduction

Let us consider an insurer (I) and a reinsurer (R) entering a reinsurance treaty related to a risk (X) for a fixed period of time. To be realistic and interesting for both actors, such a treaty should avoid any situation where one of the parties would cover the entire claim, while the other actor can make risk-free profits. Therefore, it is necessary to define the fair joint survival function (Φ) such that each party avoids bankruptcy and that the benefits of each party are ε-comparable, where ε establishes how fair the benefit is between the insurer and reinsurer. The properties for such a fair joint survival function will be detailed.

Several criteria can be selected to optimize a reinsurance treaty. Bowers (1997) and Vadja (1962) use the variance measure. Cai and Tan (2007), Cai et al. (2008) and Lu et al. (2013) use the value at risk (VaR) and conditional tail expectation (CTE) risk measures. Also, Tan et al. (2011) use the CTE risk measure to minimize the insurer's total risk. In Kaluzka (2008), the expected utility is used, and in Arrow (1963) the expected concave utility function is considered. In Balbás et al. (2009), the authors analyze several risk functions, such as the standard deviation, the absolute deviation, and the conditional value at risk (CVaR). Chi and Tan (2011), Chi (2012) and Chi and Tan (2013) also consider the CVaR as well as the VaR measures, under the expected value principle, variance related premium principles, and general premium principles. Zhu (2013) uses the Haezendonck risk measure to minimize the risk of the insurer. Cheung et al. (2014) consider the CTE and VaR measures as well as law-invariant convex risk measures. Cui et al. (2013) and Assa (2015) look at the distortion risk measure under general premium principle, including the expected value and Wang's premium principle and distortion risk premium principle. All these studies consist in looking at the insurer's—or sometime to the reinsurer's, as in Vadja (1962) and Hürlimann (2011)—point of view only. Using this perspective implies that a reinsurance treaty can be optimal for an actor (generally the insurer) while being unfavorable for the other one. Borch (1960) is the first to consider the interests of both parties, and Borch (1969) suggested that a reinsurance contract might be optimal for the insurer without being acceptable for the reinsurer.

In Cai et al. (2013), a reciprocal approach is presented, providing retention amounts, based on the joint survival or profitable probability for several reinsurance treaties. The authors recognize that their results sometimes lead to unfair situations (p.158). This is precisely why we use a refined, objective function, related to the joint survival and profitability functions that we optimize in this paper. This approach allows us to avoid these unfair cases. We aim at providing a fair share of benefits between the insurer and reinsurer. We will focus on two types of contract, quota-share and the stop-loss models, using the expected value principle for the reinsurance premiums. Fang and Qu (2012) also consider these two types of contract under the same principle. In Castañer and Bielsa (2014), only the stop-loss reinsurance contract is considered. Fang and Qu (2012) and Castañer and Bielsa (2014) aim at maximizing the joint survival function in regards to both actors' interests. In our research, we go further by introducing a modified objective function that combines the survival of both parties with a fair share of benefits, for each type of contract. Balbás et al. (2013) use deviation measures and coherent risk measures for developing risk-sharing strategies. However, the authors acknowledge that under particular conditions, the selected risk-sharing plan might provoke a high probability of global bankruptcy (p. 55). Here, we develop an objective function that we optimize, ensuring that each party avoids bankruptcy.

Let us denote by P_I and P_R the net insurance premiums received by the insurer and the reinsurer, respectively. Also, let I_f and f represent the retained loss function (i.e., the loss covered by the insurer) and the ceded loss function (i.e., the reinsurer's part covered), respectively. Finally, u_I and u_R represent the initial wealth of the insurer and the reinsurer, respectively. We have that $I_f(X) + f(X) = X$, where X represents the loss. When both parties avoid bankruptcy, then $I_f(X) \leq P_I + u_I$ and $f(X) \leq P_R + u_R$. Also, the benefits of each party being ε-comparable is equivalent to $P_I - I_f \leq \varepsilon(P_R - f)$, which can be interpreted as a way to fairly share the benefits between the insurer and reinsurer. We seek the highest probability so that both the insurer and the reinsurer avoid bankruptcy while at the same time sharing ε-fairly the benefits. In this paper, we look at both actors' interests and develop a new objective function, Φ, the fair joint survival function.

In Section 2, we expand the equations leading to the characterization of the optimization problem under the expected value principle. In Section 3, we evaluate both Φ_ε^* and f^*, representing the optimal fair joint survival function and the optimal ceded loss function, respectively, considering the quota-share reinsurance model. In Section 4, we analyze the properties of the stop-loss reinsurance contract applied to this fair joint survival function. Section 5 concludes.

2. Optimal fair joint survival function under the expected value principle

In this section, we develop a fair joint survival function (Φ) that guarantees to avoid bankruptcy for both the insurer and the reinsurer, while sharing fairly the benefits from this transaction. To ensure such conditions, we introduce a parameter (ε) that determines how fair the contract is between the insurer and the reinsurer. This parameter may vary depending on the type of the contract. We define the fair joint survival function by

$$\Phi_\varepsilon(f) = Pr\left\{\begin{matrix} P_I - I_f \leq \varepsilon(P_R - f), \\ I_f \leq P_I + u_I,\ f \leq P_R + u_R \end{matrix}\right\}.$$

Therefore, we seek to solve $\max_{f\in\mathfrak{F}} \Phi_\varepsilon(f)$ where $\mathfrak{F}$ is the set of eligible functions f defining the contract between the insurer and the reinsurer.

Then, our objective is to define the optimal contract for each type of reinsurance model. We characterize the optimal solution (f^*) and the optimal value (Φ^*) of the fair joint survival function. This function can be interpreted as an objective function and a probability function, resulting in a distribution function.

In this paper, we chose to consider the expected value principle for the reinsurance premiums, applied to two types of contract: the quota-share and the stop-loss models. Under this principle, the premiums can be formulated as

$$P_I = (1+\theta_I)E\left[I_f(X)\right] \quad \text{and}$$
$$P_R = (1+\theta_R)E[f(X)], \tag{1}$$

where P_I and P_R are respectively the insurer and the reinsurer net premium, and $\theta_R > 0$ and $\theta_I > 0$ are their relative safety loadings. For a matter of simplification, we avoid the case where $\theta_I = \theta_R$.

In the following section, we study the quota-share case.

3. Optimization of the fair joint survival function for a quota-share reinsurance model

Quota-share reinsurance is a common contract model. The insurer cedes an agreed-on percentage of the risk it insures. Let $(1 - b)$ be this percentage, where $b \in [0, 1]$. Therefore, the retained loss function is given by

$$I_f(X) = bX, \tag{2}$$

and the ceded loss function is given by

$$f(X) = (1-b)X. \tag{3}$$

Replacing Equations (2) and (3) in (1) leads respectively to

$$P_I = (1+\theta_I)b_\mu \text{ and } P_R = (1+\theta_R)(1-b)\mu, \tag{4}$$

where μ is the expected value of the loss X.

3.1. Optimizing algorithm for the quota-share model

The optimization problem we are studying is represented by

$$\max_{b\in[0,1]} \Phi_\varepsilon(b) = Pr\left\{\begin{array}{c} P_I - I_f \le \varepsilon(P_R - f), \\ I_f \le P_I + u_I,\ f \le P_R + u_R \end{array}\right\},$$

where $\Phi_\varepsilon(b)$ is the fair joint survival function. Furthermore, we have that $u_I \ge 0$ and $u_R \ge 0$. The optimization problem is equivalent to the maximization, in terms of b, of the probabily of

$$I_f \le P_I + u_I,\ f \le P_R + u_R, \quad and$$
$$P_I - I_f \le \varepsilon(P_R - f). \tag{5}$$

Using equations (2), (3), and (4), we see that minimizing (5) is equivalent to minimizing the following inequalities, in terms of b:

$$bx \le (1+\theta_I)b\mu + u_I,$$
$$(1-b)x \le (1+\theta_R)(1-b)\mu + u_R, \quad and$$
$$x \le \frac{b\gamma_I - \varepsilon\gamma_R(1-b)}{b + b_{\varepsilon-\varepsilon}}, \tag{6}$$

where $\gamma_R = (1 + \theta_R)\mu$ and $\gamma_I = (1 + \theta_I)\mu$.

Therefore, X must obey three equations having b as unknown parameter. Having X being less than or equal to three limits is equivalent to X being less than or equal to the minimum of these limits. Let us define the limits to be respected by the loss X to ensure respectively the survival of the insurer (L_I), the survival of the reinsurer (L_R) and a fair share of the benefit (L_O) as

$$L_I = \gamma_I + \frac{u_I}{b},\ L_R = \gamma_R + \frac{u_R}{1-b}, \quad and$$
$$L_O = \frac{b\gamma_I - \varepsilon\gamma_R(1-b)}{b + b_{\varepsilon-\varepsilon}}.$$

Remark. *We set* $\varepsilon > \frac{b}{1-b}$ *to avoid problematic cases. If* $\varepsilon \le \frac{b}{1-b}$*, then the third equation in (6) would become*

$$x \ge \frac{b\gamma_I - \varepsilon\gamma_R(1-b)}{b + b_{\varepsilon-\varepsilon}},$$

which implies that Φ^* would be equal to $F(min(L_I, L_R)) - F(L_O)$. This situation results in substantially low values of Φ^*, which are undesirable situations that both the insurer and reinsurer would avoid.

The next propositions compare the three limits to find the minimum ranges of b values that would optimize the fair joint survival function. In each comparison, two cases exist, whether $\gamma = \gamma_I - \gamma_R$ is positive or negative. This option, which is equivalent to comparing (θ_I) and (θ_R), will determine the optimal value of the fair joint survival function. Figure 1 illustrates that this optimal value will be expressed in terms of the distribution function $F(x)$.

The next proposition compares L_I and L_R.

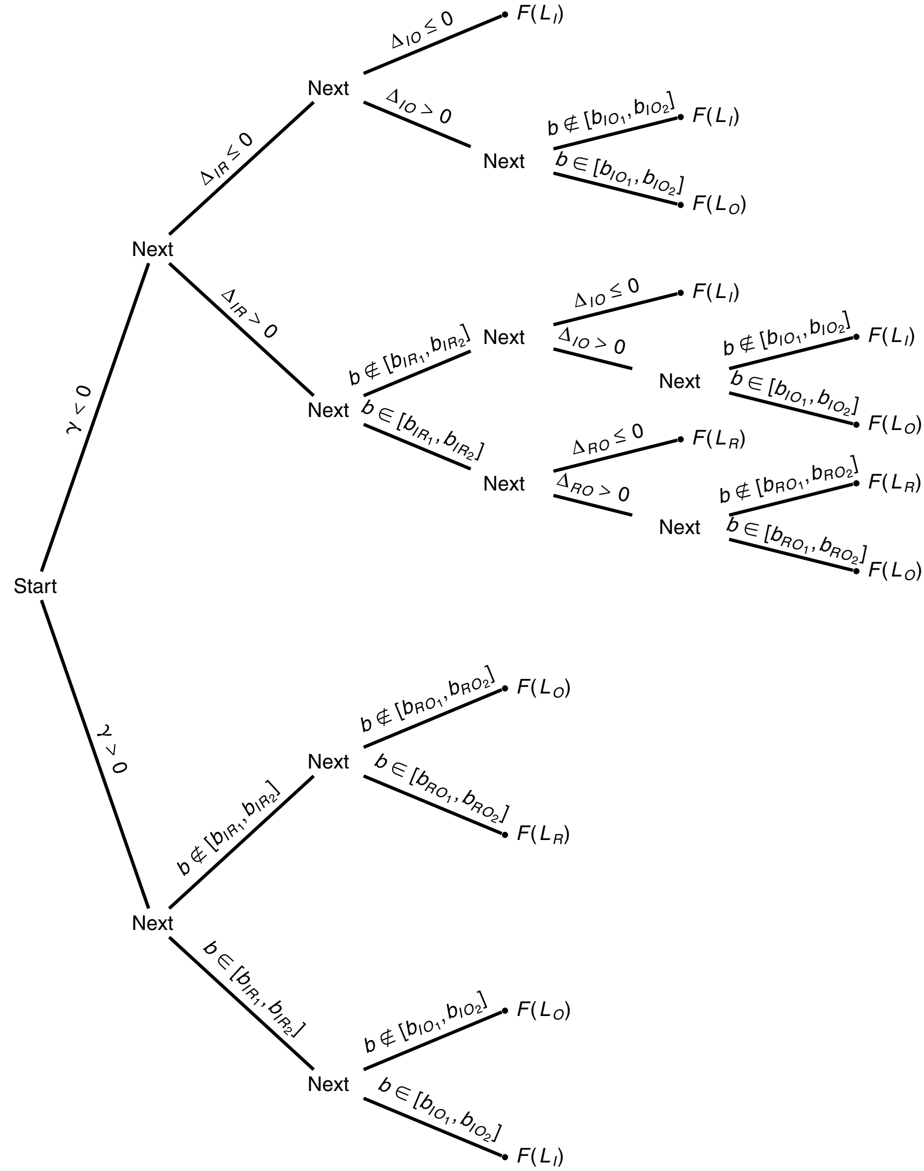
Proposition 1. *Let* $\beta_{IR} = u_R + u_I - \gamma$ *and* $\Delta_{IR} = \beta_{IR}^2 + 4\gamma u_I > 0$*. Then using* $b_{IR_1} = \frac{-\beta_{IR} - \sqrt{\Delta_{IR}}}{2\gamma}$ *and* $b_{IR_2} = \frac{-\beta_{IR} + \sqrt{\Delta_{IR}}}{2\gamma}$ *leads to two different cases. The first one concerns* $\gamma > 0$*. If* $b \in [b_{IR_1}, b_{IR_2}]$*, we have that* $L_R > L_I$*. On the other hand, when* $\gamma < 0$*, the comparison between* L_I *and* L_R *depends on the sign of* Δ_{IR} *and whether b belongs or not to* $[b_{IR_1}, b_{IR_2}]$*. Thus,* $L_I > L_R$ *if* $\Delta_{IR} > 0$ *and* $b \in [b_{IR_1}, b_{IR_2}]$.

Note that $b\varepsilon[b_{IR_1}, b_{IR_2}]$ does not ensure L_R being greater than L_I. The comparison between the safety loadings of the insurer and the reinsurer matters as much as the range of b. Consequently, the choice of θ_I and θ_R and the initial wealths u_I and u_R will determine which actor imposes the limit to be respected by the loss X so that both of them avoid bankruptcy.

The next proposition compares L_I and L_O.

Proposition 2. *Let* $\Delta_{IO} = \beta_{IO}^2 + 4\varepsilon^2\gamma u_I > 0$*, where* $\beta_{IO} = u_I + \varepsilon u_I - \varepsilon\gamma$*. Then, using* $b_{IO_1} = \frac{-\beta_{IO} - \sqrt{\Delta_{IO}}}{2\varepsilon\gamma}$ *and* $b_{IO_2} = \frac{-\beta_{IO} + \sqrt{\Delta_{IO}}}{2\varepsilon\gamma}$*, we establish that when* $\gamma > 0$*, then* $L_O > L_I$ *is equivalent to* $b \in [b_{IO1}, b_{IO2}]$.

Figure 1. Decision tree for optimizing the fair joint survival function



However, when $\gamma < 0$, then $L_I > L_O$ if and only if $b \in [b_{IO1}\ b_{IO2}]$ and $\Delta_{IO} > 0$.

Proposition 2 suggests, again, that the lower limit between L_I ensuring the survival of the insurer and L_O ensuring a fair share of the benefits depends on the value of the safety loadings difference $\theta_I - \theta_R$ and the initial wealths of the insurer and reinsurer. Once these values are set, we can determine which values b can take, depending on the comparison required between L_I and L_O.

The next proposition compares L_R and L_O.

Proposition 3. *Let* $\Delta_{RO} = \beta_{RO}^2 + 4\varepsilon\gamma u_R > 0$, *where* $\beta_{RO} = u_R + \varepsilon u_R - \gamma$. *Then, using* $b_{RO_2} = \dfrac{-\beta_{RO} - \sqrt{\Delta_{RO}}}{2\gamma}$ *and* $b_{RO_1} = \dfrac{-\beta_{RO} + \sqrt{\Delta_{RO}}}{2\gamma}$, *we establish that when* $\gamma > 0$, *then* $L_O > L_R$ *is equivalent to* $b \in [b_{RO_1}, b_{RO_2}]$.

Whereas when $\gamma < 0$, then $L_R > L_O$ if and only if $\Delta_{RO} > 0$ and $b \in [b_{RO_1}, b_{RO_2}]$.

The comparison between these limits L_R and L_O has the same interpretation as for the comparison of L_I and L_O. L_R can be greater than L_O depending on the parameters of the problem (i.e., u_I, u_R, θ_I and θ_R).

We can illustrate the resulting optimal probabilities, based on different retention level ranges. Twelve cases are derived from the initial choice of γ representing the safety loadings difference, and then on the sets of possible values of b. This means that the safety loadings of each actor directly impacts the optimal result of their fair joint survival probability, with a comparable profit. More precisely, the comparison between these safety loadings is the starting point for the initial branches of the tree summarizing the algorithm produced in this section.

To illustrate the previous propositions leading to the decision tree, we here provide a simple illustration.

Example 1. *Consider a loss X following a compound Poisson distribution with average $E[X] = 2000$. We set the safety loadings $\theta_I = 0.04$ and $\theta_R = 0.02$, the initial wealths $u_I = 300$ and $u_R = 1500$, and $\varepsilon = 0.7$. We set a hypothetic value to the retention $b = 0.4$ in order to follow the paths in Figure 1.*

Browsing the decision tree and noting that $\gamma > 0$, $b \notin [b_{IR_1}; b_{IR_2}] = [-44.17;0.17]$ and $b \in [b_{RO1}; b_{RO2}] = [-63.17;0.42]$, we conclude that $\Phi_\varepsilon^*(b) = F(L_R(b)) = 0.9990071$.

For purpose of verification, we check that $\varepsilon > \frac{b}{1-b}$, $F(L_R(b)) > F(L_I(b))$ and $F(L_R(b)) > F(L_O(b))$.

In Example 1, we set b to illustrate the algorithm and obtain $\Phi^*(b) = 0.9990071$. The following section provides more realistic scenarios, optimizing the retention level to obtain maximal fair joint function values.

3.2. Optimal value of the quota-share retention

In this section, we discuss how we can evaluate the optimal quota-share retention b^* such that $f^*(x) = (1 - b^*)x$ defines the quota-share reinsurance contract that optimizes the fair joint survival function between the insurer and the reinsurer.

First, the optimal value b^* depends on the distribution function $F(x)$. Then, several properties can be summarized in the following proposition.

Proposition 4. *The maximization of $\Phi_\varepsilon(b)$ depends on the minimum of L_I, L_R and L_O. Maximizing $F(L_I(b))$ is equivalent to minimizing b on the subset, where $L_I = \min(L_I, L_R, L_O)$. Also, maximizing $F(L_R(b))$ is equivalent to maximizing b on the subset where $L_R = min(L_I,L_R,L_O)$. Finally, maximizing $F(L_O(b))$ is equivalent to minimizing b if $\gamma > 0$ and maximizing b if $\gamma < 0$, on the subset where $L_O = min(L_I, L_R, L_O)$.*

Therefore, by referring to Figure 1, we can establish the following theorems.

Theorem 1

For $\gamma > 0$, i.e., when the safety loading of the insurer (θ_I) is greater than the reinsurer's (θ_R), the optimal value of the fair joint survival function and of the quota-share retention are, respectively,

$$\Phi_\varepsilon^* = \max\left\{\begin{matrix} F(L_I(\sigma_1)); F(L_O(\sigma_2)); \\ F(L_R(-\sigma_3)); F(L_O(\sigma_4)) \end{matrix}\right\}$$

and

$$b^* = \underset{\sigma_i}{\arg\max}\left\{\begin{matrix} F(L_I(\sigma_1)); F(L_O(\sigma_2)); \\ F(L_R(-\sigma_3)); F(L_O(\sigma_4)) \end{matrix}\right\},$$

where $\forall i \in [1; 4]$, $\sigma_i = \min(\Sigma_i)$ and

- $\Sigma_1 = [0; 1] \cap [b_{IR1}; b_{IR2}] \cap [b_{IO1}; b_{IO2}]$,
- $\Sigma_2 = [0; 1] \cap [b_{IR1}; b_{IR2}] \backslash [b_{IO1}; b_{IO2}]$,
- $\Sigma_3 = -\{[0; 1] \cap [b_{RO1}; b_{RO2}] \backslash [b_{IR1}; b_{IR2}]\}$,
- $\Sigma_4 = [0; 1] \backslash \{[b_{IR1}; b_{IR2}] \cup [b_{RO1}; b_{RO2}]\}$.

Example 2. *We illustrate Theorem 1 using the framework provided in Example 1. We can compare the results given by the optimization Theorem 1 and the results obtained from arbitrarily chosen values. Here, we want to obtain b, representing the unknown retention level. According to Theorem 1, we first need to evaluate the sets Σ_i and then determine $\sigma_i = min(\Sigma_i)$. A quick computation leads to the following values,*

	Σ_i		σ_i
Σ_1	[0; 0.1698]	σ_1	0
Σ_2	Ø	σ_2	Ø
Σ_3	[–0.4156; –0.1698]	σ_3	–0.4156
Σ_4	[0.4156; 1]	σ_4	0.4156

The next step is to evaluate $F(L_I(\sigma_1))$, $F(L_O(\sigma_2))$, $F(L_R(-\sigma_3))$ and $F(L_O(\sigma_4))$. We obtain $F(L_R(-\sigma_3))$ = 0.9991195 as a maximum value. Finally, we conclude that the optimal retention value $b^* = -\sigma_3 = 0.4156$ and $\Phi^*_\varepsilon = F(L_R(-\sigma_3)) = 0.999195$.

For negative values of γ, we refer to the upper half of Figure 1. We will use the symbol $S \cap UC_{\{condition^*\}}$ to denote the subset S under *condition**. If *condition** is not respected, then $S \cap UC_{\{condition^*\}} = \varnothing$.

Theorem 2

For $\gamma < 0$, i.e., when the safety loading of the insurer (θ_I) is lower than the reinsurer's (θ_R), the optimal value of the fair joint survival function and of the quota-share retention are, respectively,

$$\Phi^*_\varepsilon = \max\begin{Bmatrix} F(L_O(-\pi_1)); F(L_R(-\pi_2)); F(L_R(-\pi_3)); \\ F(L_O(-\pi_4)); F(L_I(\pi_5)); F(L_I(\pi_6)); \\ F(L_O(-\pi_7)); F(L_I(\pi_8)); F(L_I(\pi_9)) \end{Bmatrix}$$

and

$$b^* = \underset{\pi_i}{\operatorname{argmax}}\begin{Bmatrix} F(L_O(-\pi_1)); F(L_R(-\pi_2)); F(L_R(-\pi_3)); \\ F(L_O(-\pi_4)); F(L_I(\pi_5)); F(L_I(\pi_6)); \\ F(L_O(-\pi_7)); F(L_I(\pi_8)); F(L_I(\pi_9)) \end{Bmatrix},$$

where $\forall i \in [1; 9]$, $\pi_i = min(\Pi_i)$ such that:

- $\Pi_1 = -\{[0; 1] \cap [b_{IR1}; b_{IR2}] \cap [b_{RO1}; b_{RO2}]\} \cap UC_{\{\Delta_{IR}>0;\ \Delta_{RO}>0\}}$,
- $\Pi_2 = -\{[0; 1] \cap [b_{IR1}; b_{IR2}] \backslash [b_{RO1}; b_{RO2}]\} \cap UC_{\{\Delta_{IR}>0;\ \Delta_{RO}>0\}}$,
- $\Pi_3 = -\{[0; 1] \cap [b_{IR1}; b_{IR2}]\} \cap UC_{\{\Delta_{IR}>0;\ \Delta_{RO}\le 0\}}$,
- $\Pi_4 = -\{[0; 1] \cap [b_{IO1}; b_{IO2}] \backslash [b_{IR}; b_{IR2}]\} \cap UC_{\{\Delta_{IR}>0;\ \Delta_{IO}>0\}}$,
- $\Pi_5 = \{[0; 1] \backslash \{[b_{IR1}; b_{IR2}] \cup [b_{IO1}; b_{IO2}]\}\} \cap UC_{\{\Delta_{IR}>0;\ \Delta_{IO}>0\}}$,
- $\Pi_6 = \{[0; 1] \backslash [b_{IR1}; b_{IR2}]\} \cap UC_{\{\Delta_{IR}>0;\ \Delta_{IO}>0\}}$,
- $\Pi_7 = -\{[0; 1] \cap [b_{IO1}; b_{IO2}]\} \cap UC_{\{\Delta_{IR}\le 0;\ \Delta_{IO}>0\}}$,
- $\Pi_8 = \{[0; 1] \backslash [b_{IO1}; b_{IO2}]\} \cap UC_{\{\Delta_{IR}\le 0;\ \Delta_{IO}>0\}}$,
- $\Pi_9 = [0; 1] \cap UC_{\{\Delta_{IR}\le 0;\ \Delta_{IO}\le 0\}}$.

Example 3. *To illustrate Theorem 2, we assume that X follows a Pareto distribution with parameters* x_m = 1 *and* $k = 2$. *We set* $\theta_I = 0.02$, $\theta_R = 0.04$, $u_I = 5$, $u_R = 10$ *and* $\varepsilon = 1.2$. *We obtain the following intermediary results:*

Variables	Value
γ	–0.04
Δ_{IR}	225.4
Δ_{IO}	120.9
Δ_{RO}	483.8
$[b_{IR_1}; b_{IR_2}]$	[0.33;375.7]]
$[b_{IO_1}; b_{IO_2}]$	[0.54;229.6]
$[b_{RO_1}; b_{RO_2}]$	[0.54;550.5]

Π_i		π_i	
Π_1	[–0.54; –1]	π_1	–1
Π_2	[–0.33; –0.54]	π_2	–0.54
Π_3	Ø	π_3	Ø
Π_4	Ø	π_4	Ø
Π_5	[0; 0.33]	π_5	0
Π_6	[0; 0.33]	π_6	0
Π_7	Ø	π_7	Ø
Π_8	Ø	π_8	Ø
Π_9	Ø	π_9	Ø

Consequently, $F(L_O(-\pi_1)) = 0.7597078$ and $F(L_R(-\pi_2)) = 0.9982374$ must be compared. We find that $b^* = -\Pi_2 = 0.54$ is the optimal retention value and $\Phi^*_\varepsilon = F(L_R(-\pi_2)) = 0.9982374$ is the optimal fair joint probability value.

Theorem 1 and Theorem 2 can easily be derived from the algorithm presented in Figure 1.

In this section, we highlighted the importance of the optimal retention and the optimal value of the fair joint survival function. This optimal value is defined by $F(L_I(b^*))$, $F(L_R(b^*))$ and $F(L_O(b^*))$. Consequently, one can define, for a limited set of values for ε, a fair reciprocal treaty under a quota-share contract. We will now consider the stop-loss contract.

4. Optimization of the fair joint survival function for a stop-loss reinsurance model

The stop-loss treaty is a non-proportional type of contract where the ceded loss function is represented by $f(X) = max\{X - d; 0\}$ and the retained loss function is given by $I_f(X) = min\{X; d\}$. Consequently, knowing that $E[I_f(X)] = E[X - f(X)]$, Equation (1) can be respectively expressed by $P_R(d) = (1 + \theta_R)\int_d^{\infty} S(x)\,dx$, and $P_I(d) = \gamma_I - \frac{1+\theta_I}{1+\theta_R}P_R d$, where $\gamma_I = (1 + \theta_I)\mu$ and S is the survival function. The optimization of the fair joint survival function applied to a stop-loss is presented in what follows.

4.1. Optimal retention and fair joint survival function

In this section, we provide the conditions under which a fair reciprocal treaty under a stop-loss contract type exists. We also evaluate the optimal retention of this treaty and the optimal value of the fair joint survival function.

Theorem 3. *For a stop-loss contract with $d \geq 0$, consider the following assumptions:*

- **H1**: $d + (1 + \theta_I)\int_d^{\infty} S(x)\,dx - \gamma_I - u_I = 0$ has a unique solution $\tilde{d}$,
- **H2**: $d + (1 + \theta_I)\int_d^{\infty} S(x)\,dx - \gamma_I - \varepsilon u_R = 0$ has a unique solution $\hat{d}$,
- **H3**: $\varepsilon < \frac{u_I}{u_R}$,
- **H4**: $\frac{\varepsilon\theta_R + \theta_I}{\varepsilon + 1} \leq \frac{F(\dot{d})}{1 - F(\dot{d})}$, where $\dot{d} = min\{\tilde{d}, \hat{d}\}$.

If **H1–H4** hold, then the optimal retention d^* and the optimal fair joint survival function $\Phi_\varepsilon^* = \Phi_\varepsilon(d^*)$ exist and are defined as

$$\begin{cases} d^* = \tilde{d} \text{ and } \Phi_\varepsilon^* = F\left(P_R(\tilde{d}) + u_R + \tilde{d}\right) & \text{if } \theta_R \leq \dfrac{F(\hat{d})}{1 - F(\hat{d})}, \\ d^* = \hat{d} \text{ and } \Phi_\varepsilon^* = F\left(P_R(\hat{d}) + u_R + \hat{d}\right) & \text{if } \theta_R > \dfrac{F(\tilde{d})}{1 - F(\tilde{d})}. \end{cases}$$

Example 4. *To illustrate Theorem 3, we assume that X follows an exponential distribution with E[X] = 2. We set $\theta_I = 0.02$, $\theta_R = 0.04$, $u_I = 10$, $u_R = 5$ and $\varepsilon = 0.9$, hence $\gamma_I = 10.2$ Because $\int_d^{\infty} S(x)\,dx = \frac{exp(-\lambda d)}{\lambda}$, $d \gg (1 + \theta_I)\int_d^{\infty} S(x)dx$ when $d \geq 5$. Then, $\tilde{d} = \gamma_I - u_I = 12.04$ is the unique solution of equation **H1**. Similarly, $\hat{d} = \gamma_I - \varepsilon u_R = 6.54$ is the unique solution of equation **H2**. As $\dot{d} = min\{\tilde{d}, \hat{d}\} = 6.54$, then $F(\dot{d}) = 0.9619936$ and thus hypotheses **H3** and **H4** hold. Finally, $\theta_R \leq \frac{F(\hat{d})}{1 - F(\hat{d})}$. Then the optimal retention is $d^* = \tilde{d} = 12.04$ and the optimal fair joint probability is $\Phi_\varepsilon^* = F(P_R(\tilde{d}) + u_R + \tilde{d}) = 0.9975703$.*

Under certain hypotheses, we can define a fair reciprocal treaty between the insurer and the reinsurer based on a stop-loss contract. The optimal value of the retention and the optimal value of the fair joint survival function depend on ε, the variable defining the agreement between the two actors. This variable itself depends, for its admissible bound, on the initial wealths (u_I and u_R) of the insurer and reinsurer. The larger $\frac{u_I}{u_R}$ is, the more flexibility ε has.

5. Conclusion

In this paper, we introduce a new method to obtain a balanced joint survival and fair joint profitability between an insurer and a reinsurer for quota-share and stop-loss reinsurance treaties, under the expected value principle. Our motivation is to develop such a reciprocal reinsurance treaty that would, at the same time, be optimal and in the best interest of both stakeholders. Usually, research that focuses on these reinsurance treaties only considers the insurer's point of view, leading to an unacceptable situation for the reinsurer, where the reinsurer can become bankrupt while the insurer gains all the benefits. Here, we use a fairness variable (ε), which can vary depending on the actors' agreement and under certain conditions. We restricted our study to contracts under the expected value principle. Other models than quota-share and stop-loss could be considered, and other pricing principles.

Acknowledgments

This research was funded by Concordia University and the Natural Sciences and Engineering Research Council of Canada.

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A. Proofs

A.1. Proof of Proposition 1

Comparing L_I and L_R, we have that

$$L_I - L_R = 0 \Leftrightarrow \gamma b^2 + (u_R + u_I - \gamma)b - u_I = 0, \quad (7)$$

where $\gamma = \gamma_I - \gamma_R$. If $\gamma > 0 \Leftrightarrow \theta_I > \theta_R$, then (7) can take three schemes, as illustrated in Figure 2. Figure 2 shows that a curve can have zero, one, or two roots (i.e., respectively be over, reach on one unique point, or cross the X-axis).

Moreover, because $\Delta_{IR} = \beta_{IR}^2 + 4\gamma u_I > 0$, where $\beta_{IR} = u_R + u_I - \gamma$, only the lowest curve describes this case.

Then, using $b_{IR_1} = \dfrac{-\beta_{IR} - \sqrt{\Delta_{IR}}}{2\gamma}$ and $b_{IR_2} = \dfrac{-\beta_{IR} + \sqrt{\Delta_{IR}}}{2\gamma}$, we obtain that if $b \in [b_{IR_1}, b_{IR_2}]$, then $L_R > L_I$. Otherwise, $L_I > L_R$.

In the second case, when $\gamma < 0 \Leftrightarrow \theta_R > \theta_I$, the quadratic function (7) can take three schemes, as illustrated in Figure 3.

Therefore, if $\Delta_{IR} > 0$ and if $b \in [b_{IR_1}, b_{IR}]$ then $L_I > L_R$.

Figure 2. Three schemes of the quadratic function (7) when $\gamma > 0$

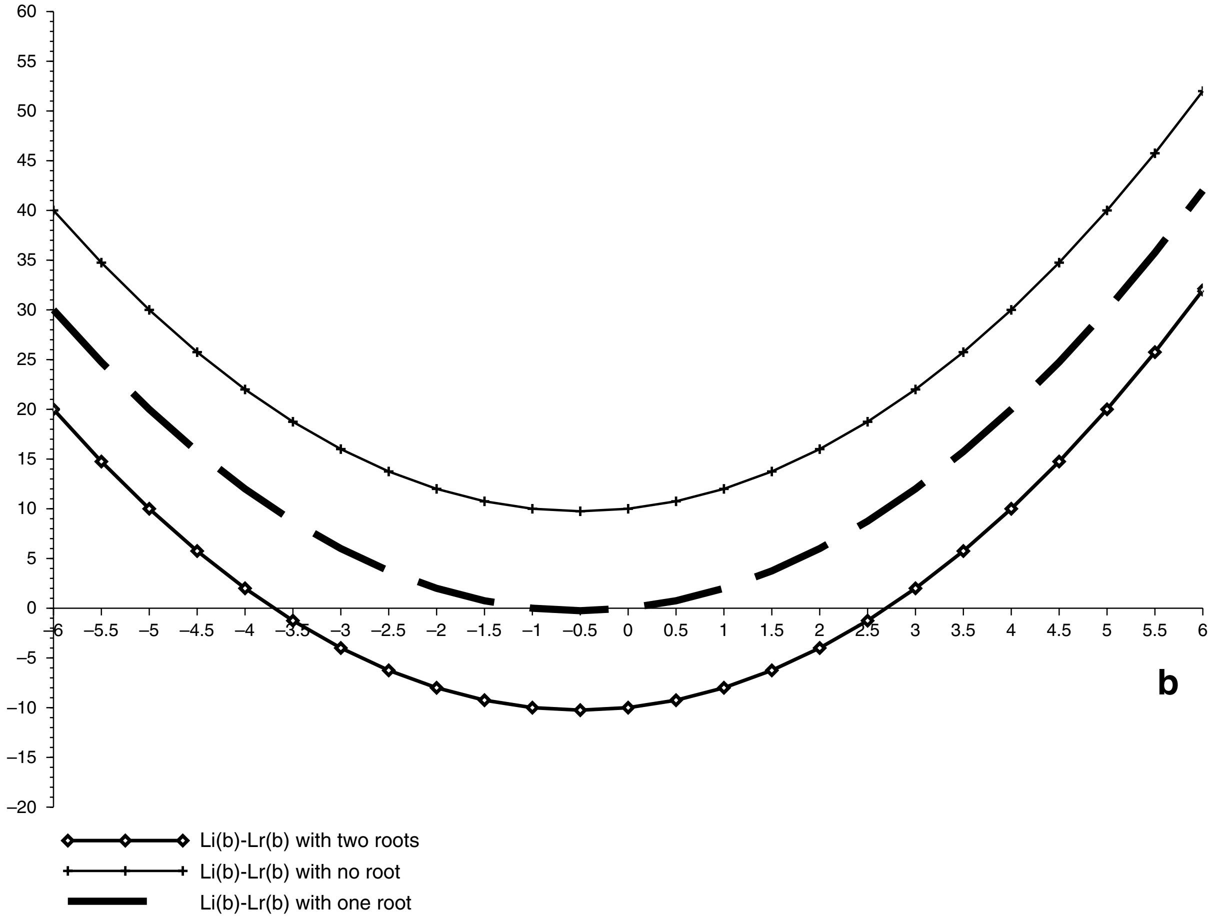
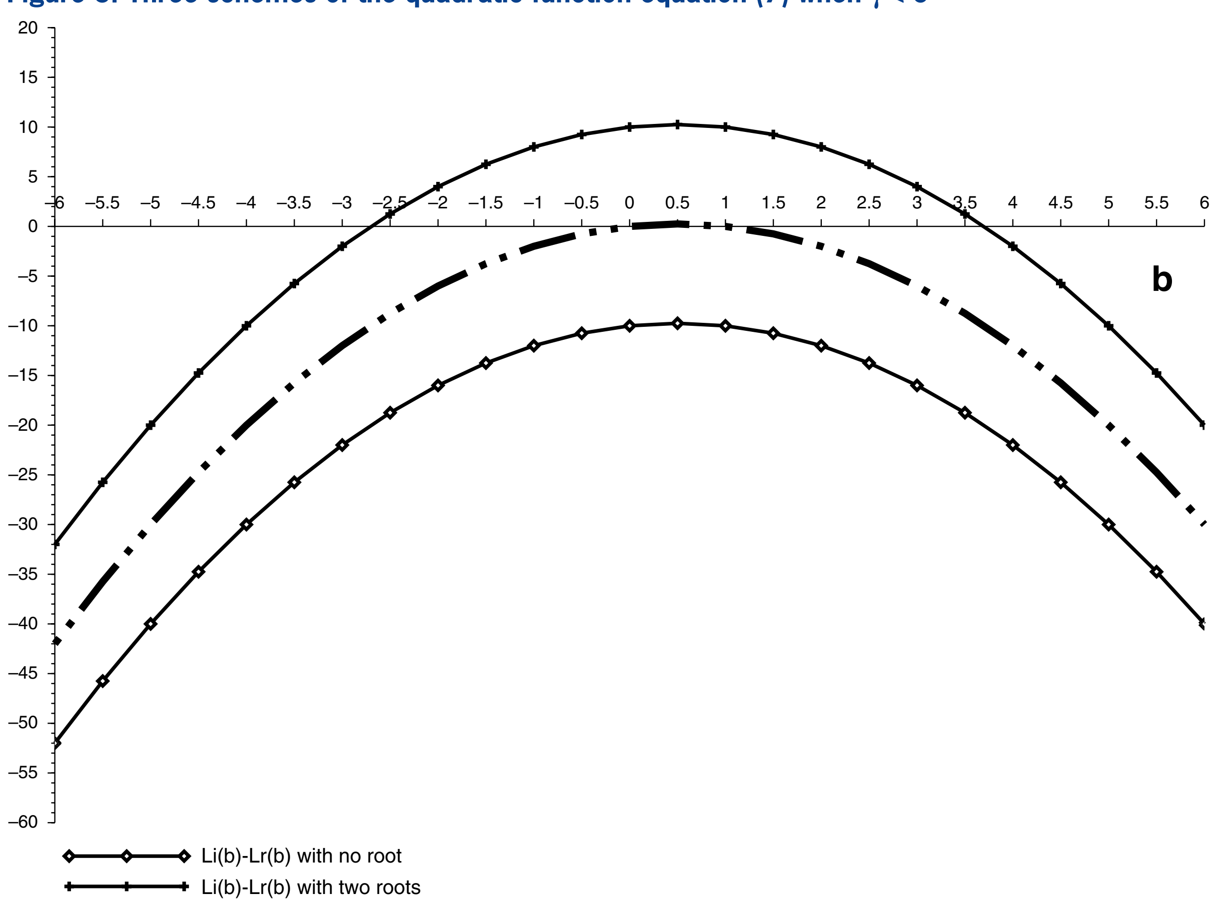


Figure 3. Three schemes of the quadratic function equation (7) when $\gamma < 0$



A.2. Proof of Proposition 2

We use the same reasoning as for the proof of Proposition 1 with

$$L_I - L_O = 0 \Leftrightarrow \gamma\varepsilon b^2 + (u_I + \varepsilon u_I - \varepsilon\gamma)b - \varepsilon u_I = 0.$$

A.3. Proof of Proposition 3

Again, the same demonstration as for the proof of Proposition 1 holds considering the following equivalence:

$$L_R - L_O = 0 \Leftrightarrow \gamma b^2 + (u_R + \varepsilon u_R + \gamma_R - \gamma_I)b - \varepsilon u_R = 0.$$

A.4. Proof of Proposition 4

We have that $F(x)$ is increasing for $x \in]0; +\infty[$, $L_I(b) = \gamma_I + \dfrac{u_I}{b}$ is decreasing for $b \in [0;1]$, $L_R(b) = \gamma_R + \dfrac{u_R}{1-b}$ is increasing for $b \in [0;1]$, and $L_O(b) = \dfrac{(\gamma_I + \gamma_R)b - \varepsilon\gamma_R}{(1+\varepsilon)b - \varepsilon}$, for $b \in [0;1]$, is increasing if $\gamma < 0$ (i.e., if $\theta_I < \theta_R$, respectively the insurer and reinsurer safety loadings); and decreasing if $\gamma > 0$ (i.e., $\theta_I > \theta_R$).

A.5. Proof of Theorem 3

We have that $f(X) = max\{X - d, 0\}$, $I_f(X) = min\{X, d\}$, $P_I = (1+\theta_I)E[I_f(x)]$, $P_R = (1+\theta_R)E[f(x)]$ and $d \in \mathbb{R}^+$. To solve $\max\limits_{d\in\mathbb{R}^+} \Phi_\varepsilon(d) = Pr\{I_f \le P_I + u_I;$ $f \le P_R + u_R;\ P_I - I_f \le \varepsilon\,(P_R - f)\}$, where $\Phi_\varepsilon(d)$ is the fair joint survival function, one must maximize, in terms of d, the probability of

$$\begin{cases} I_f \le P_I + u_I, \\ f \le P_R + u_R, \\ P_I - I_f \le \varepsilon(P_R - f), \end{cases} = \begin{cases} \min\{X,d\} \le P_I + u_I, \\ \max\{X-d,0\} \le P_R + u_R, \\ P_I - \min\{X,d\} \le \\ \quad \varepsilon(P_R - \max\{X-d,0\}). \end{cases}$$

Therefore, the fair joint survival function can expressed by

$$\begin{aligned}\Phi_\varepsilon(d) &= Pr\{X \le d\}Pr\{X \le P_I + u_I, 0 \le P_R + u_R, \\ &\quad P_I - X \le \varepsilon P_R | X \le d\} + Pr\{X > d\}Pr\{d \le P_I + u_I, \\ &\quad X - d \le P_R + u_R, P_I - d \le \varepsilon(P_R - X + d) | X > d\}.\end{aligned}$$

Consequently,

$$\Phi_\varepsilon(d) = \begin{cases} Pr\{X \le P_I + u_I, X \ge P_I - \varepsilon P_R\} & \text{if } d > P_I + u_I, \\ Pr\left\{X \le P_R - \dfrac{P_I}{\varepsilon} + \left(1 + \dfrac{1}{\varepsilon}\right)d, X \le d + P_R + u_R\right\} & \text{if } d \le P_I + u_I. \end{cases} \tag{8}$$

To evaluate the second case in (8), we need to compare the two limits that X must respect. This is equivalent to

$$P_R - \frac{P_I}{\varepsilon} + \left(1 + \frac{1}{\varepsilon}\right)d \ge d + P_R + u_R \Leftrightarrow d \ge P_I + \varepsilon u_R.$$

Thus, the fair joint survival function is expressed by

$$\Phi_\varepsilon(d) = \begin{cases} F(P_I + u_I) - F(P_I - \varepsilon P_R) & \text{if } d > P_I + u_I, \\ F\left(P_R - \dfrac{P_I}{\varepsilon} + \left(1 + \dfrac{1}{\varepsilon}\right)d\right) & \text{if } d \le min\{P_I + u_I, P_I + \varepsilon u_R\}, \\ F(d + P_R + u_R) & \text{if } d \in [P_I + \varepsilon u_R; P_I + u_I]. \end{cases} \tag{9}$$

The last case of equation (9) exists if and only if $P_I + u_I > P_I + \varepsilon u_R$, which is equivalent to hypothesis **H1** $\left(\varepsilon < \dfrac{u_I}{u_R}\right)$.

By considering (9), $\Phi_\varepsilon(d)$ can take three expressions (Φ^1, Φ^2 and Φ^3), depending on the values of d respectively on $]P_I + u_I; +\infty[$, $[0; \min\{P_I + u_I, P_I + \varepsilon u_R\}]$ and $[P_I + \varepsilon u_R; P_I + u_I]$. We have that

$$\Phi^1(d) = F(P_I(d) + u_I) - F(P_I(d) - \varepsilon P_R(d)), \tag{10}$$

$$\Phi^2(d) = F\left(P_R(d) - \frac{P_I(d)}{\varepsilon} + \left(1 + \frac{1}{\varepsilon}\right)d\right), \quad (11)$$

and

$$\Phi^3(d) = F(d + P_R(d) + u_R). \quad (12)$$

To maximize $\Phi^1(d)$ defined by equation (10), we use

$$\begin{aligned} P_R(d) &= (1+\theta_R)E[f(X)] \\ &= (1+\theta_R)E[max\{X-d;0\}] \\ &= (1+\theta_R)\int_d^\infty S(x)\mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} P_I(d) &= (1+\theta_I)E\left[I_f(X)\right] \\ &= (1+\theta_I)E[E-f(X)] \\ &= (1+\theta_i)\mu - \frac{1+\theta_I}{1+\theta_R}P_R(d) \\ &= \gamma_I - \frac{1+\theta_I}{1+\theta_R}P_R(d). \end{aligned}$$

With the differentiation formula for $h(x) = \int_{a(x)}^\infty f(x, y)\,dy$, that is given by

$$\frac{dh(x)}{dx} = \int_{a(x)}^\infty \frac{\partial f(x,y)}{\partial x}\mathrm{d}y - f(x, a(x))\frac{da(x)}{dx},$$

we obtain

$$P_R'(d) = -(1+\theta_R)S(d) \quad (13)$$

and

$$P_I'(d) = (1+\theta_I)S(d). \quad (14)$$

Hence,

$$[P_I(d) + u_I]' = (1+\theta_I)S(d)$$

and

$$[P_I(d) - \varepsilon P_R(d)]' = [(1+\theta_R)\varepsilon + (1+\theta_I)]S(d).$$

We can conclude that $[P_I(d) - \varepsilon P_R(d)]' > [P_I(d) + u_I]'$ which means that the increase of $[P_I(d) - \varepsilon P_R(d)]$ is greater than the increase of $[P_I(d) + u_I]$. Since $F(x)$ is an increasing function, the increase of $F(P_I(d) - \varepsilon P_R(d))$ is greater than the increase of $F(P_I(d) + u_I)$.

Thus, $\Phi^1(d)$ is decreasing, and because $\Phi^1(d)$ is defined on the open ball $]P_I + u_I; +\infty[$, it never reaches its maximum, i.e., $\max\Phi^1(d)$ has no solution.

Secondly, to maximize $\Phi^2(d)$ defined by (11) we use $H(d) = P_R(d) - \frac{P_I(d)}{\varepsilon} + \left(1 + \frac{1}{\varepsilon}\right)d$. According to (13) and (14), we have that $H(d)' = \left(1 + \theta_R + \frac{1+\theta_I}{\varepsilon}\right)(F(d) - 1) + \frac{1+\varepsilon}{\varepsilon}$.

Therefore,

$$H(d)' \geq 0 \Leftrightarrow \frac{\varepsilon\theta_R + \theta_I}{\varepsilon + 1} \leq \frac{F(d)}{1 - F(d)}.$$

One can easily verify that $\frac{F(d)}{1-F(d)}$ is an increasing function of d, $\forall d \geq 0$. Because $\Phi^2(d)$ is defined for $d \in [0; \dot{d}]$, where $\dot{d} = min\{\hat{d}, \tilde{d}\}$, then $H(d)$ and $F(H(d)) = \Phi^2(d)$ are increasing $\forall d \in [0; \dot{d}]$ if and only if $\frac{\varepsilon\theta_R + \theta_I}{\varepsilon + 1} \leq \frac{F(0)}{1 - F(0)}$, which is impossible since $F(0) = 0$ and $\frac{\varepsilon\theta_R + \theta_I}{\varepsilon + 1} > 0$. Moreover $H(d)$ and $F(H(d)) = \Phi^2(d)$ are decreasing $\forall d \in [0; \dot{d}]$ if and only if $\frac{\varepsilon\theta_R + \theta_I}{\varepsilon + 1} \leq \frac{F(\dot{d})}{1 - F(\dot{d})}$, which contradicts hypothesis **H4**. Therefore $\Phi^2(d)$ has no optimum.

Finally, to maximize $\Phi^3(d)$ defined by (12), we use $G(d) = P_R(d) + u_R + d$ and $G(d)' = (1 + \theta_R)F(d) - \theta_R$. Therefore, $G(d)' \geq 0 \Leftrightarrow \theta_R \leq \frac{F(d)}{1-F(d)}$, where $d \in [\hat{d}; \tilde{d}]$. Because $\frac{F(d)}{1-F(d)}$ is an increasing function of d, we conclude that $G(d)$ is increasing, and thus $\Phi^3(d) = F(G(d))$ is increasing, on $[\hat{d}; \tilde{d}]$, if and only if $\theta_R \leq \frac{F(\tilde{d})}{1-F(\tilde{d})}$; then $\Phi_\varepsilon^* = \max\Phi^3(d) = F(P_R(\tilde{d}) + u_R + \tilde{d})$ and $d^* = \tilde{d}$. On the other hand, $G(d)$ is decreasing, which means that $\Phi^3(d) = F(G(d))$ is decreasing on $[\hat{d}; \tilde{d}]$, if and only if $\theta_R > \frac{F(\tilde{d})}{1-F(\tilde{d})}$. Then, $\Phi_\varepsilon^* = \max\Phi^3(d) = F(P_R(\hat{d}) + u_R + \hat{d})$ and $d^* = \hat{d}$.