Analysis of bivariate excess losses

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Abstract

The concept of excess losses is widely used in reinsurance and retrospective insurance rating. The mathematics related to it has been studied extensively. However, it seems that the formulas for higher moments of the excess losses are not readily available in the property and casualty actuarial literature. Therefore, in the first part of this paper, we introduce a formula for calculating the higher moments, based on which it is shown that they can be obtained directly from the Table of Insurance Charges (Table M). In the second part of the paper, we introduce the concept of bivariate excess losses. It is shown that the joint moments of bivariate excess losses can be computed through methods similar to the ones used in the univariate case. In addition, we provide examples to illustrate possible applications of bivariate excess loss functions.

Keywords: Moments of excess losses; Bivariate excess loss functions; Table M.

1 Introduction

The concept of excess losses is widely used in reinsurance and retrospective insurance rating. The mathematics of it has been studied extensively in the property and casualty insurance literature. See for example, Lee (1988) and Halliwell (2012). The first moment of the excess losses has been tabulated into the Table of Insurance Charges (Table M) for use in NCCI retrospective rating plan. Higher moments of excess losses can be used to measure the volatility of excess losses. However formulas for them are not readily available in the property casualty actuarial literature. One could refer to Section 2 of Miccolis (1977) for some discussions. In fact, the formulas for calculating higher moments of excess losses do exist in the literature of stochastic orders, where the nth moment of excess losses is named the nth order stop–loss transform (see for example, Hürlimann, 2000). Therefore, in the first part of this paper, we introduce the simple formulas for calculating higher moments of the excess losses to the property casualty actuarial literature. More importantly, using

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a detailed numerical example, we show that the higher moments can be obtained directly from Table M.

In the second part of this paper, we introduce the concept of bivariate excess losses, which has its origin in the reliability theory literature. See for example Zahedi (1985) and Gupta and Sakaran (1998). In the context of stochastic ordering, Denuit et. al. (1998) presented a formula for the joint moments of multivariate excess losses. In this paper, we show that the joint moments of bivariate excess losses can be computed through methods similar to the ones used in the univariate case. We provide examples to illustrate possible applications of bivariate excess loss functions.

The rest parts of the paper are organized as follows. Section 2 introduces formulas for higher moments of excess losses and show how they may be computed using Table M. Section 3 presents the theory of bivariate excess losses. Section 4 provides examples and Section 5 concludes. Proofs of some of the results are included in an appendix.

2 Univariate excess losses

We begin by introducing some notations and basic facts.

2.1 Preliminaries

Let X be a random loss variable taking non–negative values and have cumulative distribution function F and survival function S. Then the limited loss up to a retention level d is defined by

$$X_0^d = \begin{cases} X & \text{if } X \leq d \\ d & \text{if } X > d \end{cases} .$$

The loss in the layer (d, l) is defined by

$$X_d^l = X_0^l - X_0^d = \begin{cases} 0 & \text{if } X \leq d \\ X - d & \text{if } d < X \leq l \\ l - d & \text{if } X > l \end{cases} .$$

The excess loss over a limit d is defined by

$$X_d^\infty = (X - d)_+ = X - X_0^d = \begin{cases} 0 & \text{if } X \leq d \\ X - d & \text{if } X > d \end{cases} .$$

It is well known that the expected value of the limited loss is given by (see for example, Equation (1.6) in Lee 1988)

$$\mathbb{E}(X_0^d) = \int_0^d S(u)du. \tag{1}$$

Due to the importance of (1), A short proof of it is given in the appendix of the paper. The method used in the proof can be readily extended to the bivariate situation.

Because $X_d^l = X_0^l - X_0^d$, we have for the layered loss that,

$$\mathbb{E}[X_d^l] = \int_d^l S(u)du, \tag{2}$$

and for the excess loss,

$$\mathbb{E}[X_l^\infty] = \int_l^\infty S(u)du. \tag{3}$$

2.2 Higher moments of excess losses

Higher moments of the excess loss X_l^∞ can be obtained using the following Proposition.

Proposition 2.1 *Let*

$$R_1(l) = \mathbb{E}[X_l^\infty], \tag{4}$$

and for $i \geq 1$, let

$$R_{i+1}(l) = \int_l^\infty R_i(u)du. \tag{5}$$

Then

$$R_i(l) = \frac{1}{i!}\mathbb{E}[(X_l^\infty)^i], \quad \text{for } i \geq 1. \tag{6}$$

The proof of the proposition was obtained in Denuit et. al. (1998) and Hürlimann (2000), it is included in the appendix for the completeness of this paper.

If the distribution of the underlying loss X is known, then one could compute $\mathbb{E}\left[(X_l^\infty)^k\right]$ for any integer k using Proposition 2.1 iteratively. More importantly, we point out that since Table M in fact lists values of $R_1(l)$, one may compute $R_k(l), k > 1$ directly from it recursively, in a similar fashion as one would compute $R_1(l)$ from the survival function $S(l)$. This way, $\mathbb{E}\left[(X_l^\infty)^k\right], k \geq 1$ can be obtained directly from Table M. We next show the method with a numerical example.

Example 2.1:

Consider problem 4 of Brosius (2002). Let X represent the loss ratio for a homogeneous group of insureds and was observed to have values 30%, 45%, 45% and 120% respectively. Let $Y = X/\mathbb{E}(X)$ be the corresponding entry ratios and thus take values 0.5, 0.75, 0.75, 2. Table M constructed using the method described in Brosius (2002) gives the mean excess loss function of Y,

$$R_1(r) = \mathbb{E}[Y_r^\infty].$$

Then the second moment of the excess losses $\mathbb{E}\left[(Y_r^\infty)^2\right]$ may simply be obtained by numerically integrating $R_1(r)$ and then multiplying the result by 2. Realizing that $R_1(r)$ is piecewise linear between entry ratio values, the numerical integration is implemented by

$$R_2(r) = \sum_{k\geq 0} \frac{R_1(r+k\Delta) + R_1(r+(k+1)\Delta)}{2}\Delta,$$

where Δ is the interval between the entry ratio values.

Table 1 shows the details of the calculation. Here, the second column gives the Table M insurance charge values, the third column (R_2 in layer) corresponding an entry ratio r is calculated by $\frac{R_1(r)+R_1(r+\Delta)}{2}\Delta$, where Δ is the interval between entry ratios, which is 0.25 in the example. The fourth column ($R_2(r)$) is the cumulative summation of the third column. The fifth column is just the fourth one multiplied by 2.

This example showed the important fact that the higher moments of the excess losses can be obtained directly from Table M. No other information is needed!

Table 1: Calculating higher moments of excess losses using Table M

Entry ratio (r)	# of risks	$R_1(r)$	R_2 in layer	$R_2(r)$	$E[(Y_r^\infty)^2]$
0	0	1	0.21875	0.671875	1.34375
0.25	0	0.75	0.15625	0.453125	0.90625
0.5	1	0.5	0.1015625	0.296875	0.59375
0.75	2	0.3125	0.0703125	0.1953125	0.390625
1	0	0.25	0.0546875	0.125	0.25
1.25	0	0.1875	0.0390625	0.0703125	0.140625
1.5	0	0.125	0.0234375	0.03125	0.0625
1.75	0	0.0625	0.0078125	0.0078125	0.015625
2	1	0	0	0	0

The second moment of the layered losses $\mathbb{E}\left[(X_d^l)^2\right]$ is also of interest. We have

$$\begin{aligned}
\mathbb{E}\left[(X_d^l)^2\right] &= \mathbb{E}\left[(X_d^\infty - X_l^\infty)^2\right] \\
&= \mathbb{E}\left[(X_d^\infty)^2\right] + \mathbb{E}\left[(X_l^\infty)^2\right] - 2\mathbb{E}\left[(X_d^\infty)(X_l^\infty)\right] \\
&= \mathbb{E}\left[(X_d^\infty)^2\right] + \mathbb{E}\left[(X_l^\infty)^2\right] - 2\mathbb{E}\left[(X_d^l + X_l^\infty)(X_l^\infty)\right] \\
&= \mathbb{E}\left[(X_d^\infty)^2\right] - \mathbb{E}\left[(X_l^\infty)^2\right] - 2\mathbb{E}\left[(X_d^l)(X_l^\infty)\right]. \qquad (7)
\end{aligned}$$

The first two terms in the last line of (7) can be obtained from Table M, as shown in the previous example. The last term can again be obtained from Table M by applying Equation (12) derived in Section 3.

3 Bivariate excess losses

Let (X, Y) be a pair of random loss random variables with joint distribution function $F(x, y) = \mathbb{P}(X \leq x, Y \leq y)$ and joint survival function $S(x, y) = \mathbb{P}(X > x, Y > y)$.

Similar to formula (2) for the univariate case, we have the following Proposition, whose proof is provided in the appendix.

Proposition 3.1 *The first joint moment of the layered losses $X_{d_x}^{l_x}$ and $Y_{d_y}^{l_y}$ may be obtained by*

$$\mathbb{E}[X_{d_x}^{l_x} Y_{d_y}^{l_y}] = \int_{d_x}^{l_x} \int_{d_y}^{l_y} S(u, v) dv du. \qquad (8)$$

With this Proposition, the covariance between $X_{d_x}^{l_x}$ and $Y_{d_y}^{l_y}$ is obtained by

$$Cov(X_{d_x}^{l_x}, Y_{d_y}^{l_y}) = \int_{d_x}^{l_x} \int_{d_y}^{l_y} S(u, v) dv du - \int_{d_x}^{l_x} S_x(u) du \int_{d_y}^{l_y} S_y(v) dv, \qquad (9)$$

where S_x and S_y denote the marginal survival function of X and Y respectively. A somewhat similar formula to (9) can be found in Dhaene et. al. (1996).

As shown in Denuit et. al (1999), higher joint moments of the bivariate excess losses can be computed using the following result.

Proposition 3.2 *Let*

$$R_{11}(l_x, l_y) = \int_{l_x}^{\infty} \int_{l_y}^{\infty} S(u, v) dv du \qquad (10)$$

and for $(i, j) > (1, 1)$, let

$$R_{ij}(l_x, l_y) = \int_{l_x}^{\infty} R_{i-1,j}(u, l_y) du = \int_{l_y}^{\infty} R_{i,j-1}(l_x, v) dv.$$

Then,

$$R_{ij}(l_x, l_y) = \frac{1}{i!j!}\mathbb{E}[(X_{l_x}^{\infty})^i (Y_{l_y}^{\infty})^j]. \tag{11}$$

A proof of Proposition 3.2 is given in the appendix.

Similar to Proposition 2.1, Proposition 3.2 can be used to construct a bivariate Table M to tabulate the joint moments of the bivariate excess losses. Example 4.2 in the next Section provides an illustration.

In the rest of this section, we show that Proposition 3.2 may shed some lights on the joint moments of the amount in different layers of a random loss. To this end, setting $X = Y$, we have

$$S(u, v) = \mathbb{P}[X > u, Y > v] = \mathbb{P}[X > \max(u, v)] = S_x(\max(u, v)),$$

where $S_x(\cdot)$ denote the survival function of X. Then for two non-overlapping layers (d_1, l_1) and (d_2, l_2) of X with $d_2 \geq l_1$, we have

$$\begin{aligned}\mathbb{E}[X_{d_1}^{l_1} X_{d_2}^{l_2}] &= \int_{d_1}^{l_1}\int_{d_2}^{l_2} S(u, v)dvdu \\ &= \int_{d_1}^{l_1}\int_{d_2}^{l_2} S_x(v)dvdu \\ &= (l_1 - d_1)\mathbb{E}[X_{d_2}^{l_2}]. \end{aligned} \tag{12}$$

As a result, the covariance of $X_{d_1}^{l_1}$ and $X_{d_2}^{l_2}$ is given by

$$Cov[X_{d_1}^{l_1} X_{d_2}^{l_2}] = \left(l_1 - d_1 - \mathbb{E}[X_{d_1}^{l_1}]\right)\mathbb{E}[X_{d_2}^{l_2}], \tag{13}$$

which is Equation (39) of Miccolis (1977).

As mentioned in Section 2.2, formula (12) is useful in computing the second moment of layered losses X_d^l. In fact, applying it to (7) yields

$$\mathbb{E}\left[(X_d^l)^2\right] = \mathbb{E}\left[(X_d^{\infty})^2\right] - \mathbb{E}\left[(X_l^{\infty})^2\right] - 2(l-d)\mathbb{E}[X_l^{\infty}]. \tag{14}$$

Notice that all three terms on the right hand side of (14) can be obtained from Table M.

Another formula to compute the second moment of the layer losses is:

$$\begin{aligned}\mathbb{E}[(X_d^l)^2] &= \int_d^l \int_d^l S(u, v)dvdu \\ &= 2\int_d^l \int_d^u S(u, v)dvdu \\ &= 2\int_d^l \int_d^u S(u)dvdu \\ &= 2\int_d^l (u-d)S(u)du, \end{aligned}$$

from which we may write

$$\begin{aligned}
\mathbb{E}[(X_d^l)^2] &= 2\int_d^l (u-d)S(u)du \\
&= -2\int_d^l (u-d)dR_1(u) \\
&= -2(u-d)R_1(u)|_{u=d}^l + 2\int_d^l R_1(u)du \\
&= 2\left(R_2(l) - R_2(d) - (l-d)R_1(l)\right), \qquad (15)
\end{aligned}$$

which agrees with Equation (14).

4 Numerical examples

In this section, we present three examples. In the first one, we derive formulas for the joint moments of excess losses for a bivariate Pareto distribution. In the second one, we show that a bivaraite Table M can be constructed to tabulate the covariances between layers of losses from two lines of businesses. In the third example, we apply the formulas derived herein to study the interactions between per–occurrence and stop–losses limits when they coexist in an insurance policy.

Example 4.1: Bivariate Pareto Distribution

Following Wang (1998), assume that there exists a random parameter Λ such that for $i = 1, 2$, $X_i|\Lambda = \lambda$ are independent and exponentially distributed with rate parameter λ/θ_i. Then the conditional joint survival function of (X_1, X_2) given $\Lambda = \lambda$ is

$$S_{X_1,X_2|\Lambda=\lambda}(x_1, x_2) = e^{-\lambda(\frac{x_1}{\theta_1}+\frac{x_2}{\theta_2})}.$$

Assume that Λ follows a Gamma $(\alpha, 1)$ distribution with moment generating function $M_\Lambda(t) = (1-t)^{-\alpha}$. Then the unconditional distribution of (X_1, X_2) is a bivariate Pareto with the joint survival function

$$S(x,y) = \left(1 + \frac{x}{\theta_1} + \frac{y}{\theta_2}\right)^{-\alpha}. \qquad (16)$$

As extension of univariate Pareto distributions, bivariate Pareto distributions are useful in modelling bivariate losses with heavy tails. From the joint survival function (16), we have that

$$\mathbb{E}(X_{d_x}^{l_x} Y_{d_y}^{l_y}) = \int_{d_x}^{l_x}\int_{d_y}^{l_y}\left(1 + \frac{x}{\theta_1} + \frac{y}{\theta_2}\right)^{-\alpha} dydx$$

$$= \frac{\theta_1\theta_2}{(\alpha-1)(\alpha-2)}\left(\left(1 + \frac{d_x}{\theta_1} + + \frac{d_y}{\theta_2}\right)^{-\alpha+2} - \left(1 + \frac{l_x}{\theta_1} + + \frac{l_y}{\theta_2}\right)^{-\alpha+2}\right).$$

In addition, the following equations are easily obtained and will be used in the following.

$$\mathbb{E}(X_l^\infty) = \frac{\theta_1}{(\alpha-1)}\left(1+\frac{l}{\theta_1}\right)^{-\alpha+1},$$

$$\mathbb{E}(X_{l_x}^\infty Y_{l_y}^\infty) = \frac{\theta_1\theta_2}{(\alpha-1)(\alpha-2)}\left(1+\frac{l_x}{\theta_1}+\frac{l_y}{\theta_2}\right)^{-\alpha+2},$$

and

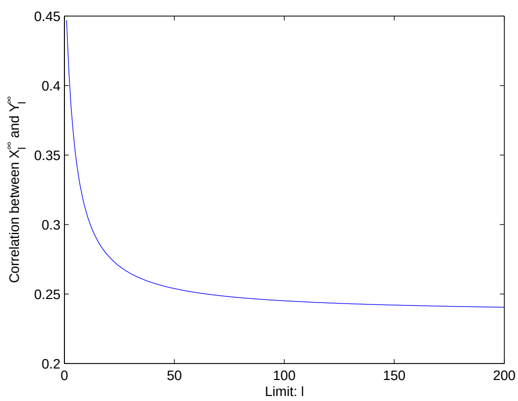
$$\begin{aligned}\mathbb{E}[(X_l^\infty)^2] &= 2\int_l^\infty (x-l)\left(1+\frac{x}{\theta_1}\right)^{-\alpha} dx \\ &= \frac{2\theta_1^2}{(\alpha-1)(\alpha-2)}\left(\frac{\theta_1+l}{\theta_1}\right)^{-\alpha+2}.\end{aligned}$$

One might wonder how the dependence between (X_l^∞) and (Y_l^∞) varies with the retention level l. For illustration, we assume that $\alpha = 3$, $\theta_1 = 5$, $\theta_2 = 10$ and calculated the correlation coefficients between X_l^∞ and Y_l^∞

$$corr(X_l^\infty, Y_l^\infty) = \frac{\mathbb{E}(X_l^\infty Y_l^\infty) - \mathbb{E}(X_l^\infty)\mathbb{E}(Y_l^\infty)}{\sqrt{Var(X_l^\infty)Var(Y_l^\infty)}}$$

for some different values of l. The relationship between the correlation coefficients and the retention level l is illustrated in Figure 1. It shows that for this particular joint distribution, the correlation coefficient decreases to some limit as the retention level l increases.

Figure 1: The correlation between X_l^∞ and Y_l^∞ as a function of l.



Example 4.2: A bivariate Table M

This example shows that a bivariate Table M can be constructed for the bivariate excess losses using a method similar to the one for constructing the univariate Table M.

Assume that one observes a sample of a pair of bivariate loss ratio random variables (X, Y) as shown in the Table 2.

X	0.6	0.8	1.2	1.4
Y	0.4	0.6	1.4	1.6

Table 2: Sample of Bivariate Loss Ratios

To compute the joint moments of the bivariate excess of losses $\mathbb{E}(X_{d_x}^{\infty} Y_{d_y}^{\infty})$, we basically need to construct their empirical joint survival function and then numerically implement the double integration in Equation (8). The detailed steps are shown in the attached Excel table. The Excel table is easy to use, for example, $\mathbb{E}[X_{d_x}^{\infty} Y_{d_y}^{\infty}]$ is simply given by the value in column J and the row with loss ratio values d_x and d_y for X and Y respectively. If it is desired to calculate the higher joint moments of $X_{d_x}^{\infty}$ and $Y_{d_y}^{\infty}$, one can proceed to do some more numerical integrations in the spreadsheet.

Example 4.3: Per-occurrence and stop–loss coverage

This example follows the one in Section 2 of Homer and Clark (2002) with some modifications. Assume that the size of Workers Compensation losses from a fictional large insured ABC follow a Pareto distribution with the survival function

$$S(x) = \left(1 + \frac{x}{\theta}\right)^{-\alpha},$$

where $\alpha = 3$ and $\theta = \$100,000$. Assume that the number of losses N follows a negative binomial distribution with the probability generating function (see for example Klugman et. al. 2012)

$$P_N(z) = (1 - \beta(z-1))^{-r},$$

where $\beta = 0.2$ and $r = 25$.

An insurance company, XYZ, has been asked to provide a per–occurrence coverage of $\$50,000$ excess of d_0 and then a stop–loss coverage on an aggregate basis of $\$500,000$ excess of d_1 .

As an actuary of XYZ, you are trying to determine an optimal combination of d_0 and d_1, so that your objective function– the ratio between the expected payments and the standard deviation of the payments, – is maximized. Notice that the expected payments can be considered as a proxy for the expected underwriting profits assuming a risk loading level, and the standard deviation of the payments of course may represent the risk level. Therefore, the objective function bears some resemblance to the Sharpe Ratio (Bodie et. al. 2009) used in portfolio analysis.

We introduce the following notations to mathematically describe the problem. The monetary unit we use is in thousands of dollars. Let the amount of a single loss be denoted by Z. Let the amount ABC has to pay per occurrence be denoted by

$$Z_A = Z_0^{d_0} + Z_{d_0+50}^{\infty}.$$

Let the amount XYZ has to pay per occurrence be denoted by

$$Z_X = Z_{d_0}^{d_0+50}.$$

Let the aggregate amount that XYZ pays for the per–occurrence coverage be denoted by

$$V = \sum_{i=1}^{N} Z_{X,i}.$$

Let the aggregate amount ABC pays after the per–occurrence coverage but before the stop–loss coverage be denoted by

$$U = \sum_{i=1}^{N} Z_{A,i}.$$

Then the total amount XYZ has to pay under the insurance treaty is given by

$$W = V + U_{d_1}^{l_1},$$

where $l_1 = d_1 + 500$.

Our goal is to select values of d_0 and d_1 so that the objective function $\mathbb{E}[W]/\sigma_W$, where σ_W stands for the standard deviation of W, is maximized.

To solve the problem, we could apply the following steps:

1. Assign some values to d_0 and d_1.

2. Construct a matrix containing the joint probability distribution function of (U, V). This can be obtained by applying the bivariate Fast Fourier Transform (FFT) method as proposed in Homer and Clark (2002).

3. Construct a matrix for the joint survival function, $S_{(U,V)}$, from the matrix for the joint probability function obtained in step 2. Construct two vectors containing values for the marginal survival functions S_U and S_V respectively.

4. Construct vectors containing values of the functions $R_1(l)$ and $R_2(l)$ for random variables U and V by applying equations (4) and (5) to the corresponding survival functions S_U and S_V. Then compute $\mathbb{E}[V]$, $\mathbb{E}[U_{d_1}^{l_1}]$, $\mathbb{E}[V^2]$, and $\mathbb{E}[\left(U_{d_1}^{l_1}\right)^2]$ using equations (6) and (15).

5. Construct a matrix containing values of the function R_{11} from $S_{(U,V)}$ using equation (10) and compute $\mathbb{E}\left[U_{d_1}^{l_1}V\right]$ by applying (11).

6. Compute the mean and the variance of $W = U_{d_1}^{l_1} + V$ using quantities obtained in steps 4 and 5; then evaluate the objective function $\frac{\mathbb{E}[W]}{\sigma_W}$.

7. Repeat steps 1–6 for different values of d_0 and d_1 and compare the values of the objective function.

Tables 3, 4 and 5 shows values of $\mathbb{E}[W]$, σ_W and the objective function $\frac{\mathbb{E}[W]}{\sigma_W}$ for some combinations of d_0 and d_1 respectively. It appears that when the per–occurrence entry point d_0 is low and the stop–loss coverage entry point d_1 is high, the objective function is maximized. In addition, the tables can be used to detect inefficient combinations of d_0 and d_1. For example, the $(d_0, d_1) = (250, 1000)$ combination results in a lower expected losses but a higher standard deviation than the $(d_0, d_1) = (200, 1500)$ combination. Therefore, the former is inefficient.

$d_0 \backslash d_1$	500	1000	1500	2000	2500
50	60.5477	51.4147	50.1536	49.8230	49.7006
100	36.9328	25.5104	24.0836	23.7256	23.5960
150	28.3191	15.3471	13.8257	13.4540	13.3209
200	24.6302	10.5683	8.9845	8.6046	8.4696
250	22.8897	8.0352	6.4053	6.0201	5.8840
300	22.0208	6.5733	4.9065	4.5175	4.3806

Table 3: The expected value of W (in thousands).

$d_0 \backslash d_1$	500	1000	1500	2000	2500
50	86.2705	56.5318	50.3110	48.4753	47.7640
100	83.5429	45.7247	36.9180	34.2140	33.1540
150	82.5044	40.6877	29.9994	26.5188	25.1145
200	81.8811	38.0508	26.0496	21.9057	20.1714
250	81.4313	36.5486	23.6461	18.9579	16.9179
300	81.0813	35.6331	22.1120	16.9900	14.6755

Table 4: The standard deviation of W (in thousands).

$d_0 \backslash d_1$	500	1000	1500	2000	2500
50	0.7018	0.9095	0.9969	1.0278	1.0405
100	0.4421	0.5579	0.6524	0.6934	0.7117
150	0.3432	0.3772	0.4609	0.5073	0.5304
200	0.3008	0.2777	0.3449	0.3928	0.4199
250	0.2811	0.2198	0.2709	0.3175	0.3478
300	0.2716	0.1845	0.2219	0.2659	0.2985

Table 5: The ratio between the mean and the standard deviation of W.

5 Conclusions

We first showed that higher moments of excess losses may be obtained from Table M. Then we showed that the joint moments of bivariate excess losses can also be obtained in a similar fashion. These techniques are useful in reinsurance and retrospective insurance rating when losses from two sources of risks are considered.

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7 Appendix

7.1 Proof of equation (1):

First of all, it is easy to verify that

$$X_0^d = \int_0^d I(X > u)du,$$

where $I(\cdot)$ is an indicator function that is equal to one when its arguments are true and zero otherwise. Then we have that

$$\mathbb{E}[X_0^d] = \mathbb{E}\left[\int_0^d I(x > u)du\right] = \int_0^d \mathbb{E}\left[I(x > u)\right]du = \int_0^d S(u)du. \quad \blacksquare$$

7.2 Proof of Proposition 2.1:

We use mathematical induction. For $i = 1$, equation (6) is true by definition. Assume that it is true for i, then

$$\begin{aligned} R_{i+1}(l) &= \int_l^\infty R_i(u)du \\ &= \int_l^\infty \frac{1}{i!}\mathbb{E}[(X-u)_+^i]du \\ &= \frac{1}{i!}\mathbb{E}\left[\int_l^\infty (X-u)_+^i du\right] \\ &= \frac{1}{(i+1)!}\mathbb{E}\left[(X-l)_+^{i+1}\right] \\ &= \frac{1}{(i+1)!}\mathbb{E}\left[(X_l^\infty)^{i+1}\right]. \quad \blacksquare \end{aligned} \tag{17}$$

7.3 Proof of Proposition 3.1:

Similar to the derivations in Section 7.1, first notice that

$$X_{d_x}^{l_x} Y_{d_y}^{l_y} = \int_{d_x}^{l_x} I(X > u)du \int_{d_y}^{l_y} I(Y > v)dv = \int_{d_x}^{l_x}\int_{d_y}^{l_y} I(X > u)I(Y > v)dvdu.$$

Then we have

$$\mathbb{E}\left[X_{d_x}^{l_x} Y_{d_y}^{l_y}\right] = \int_{d_x}^{l_x}\int_{d_y}^{l_y} \mathbb{E}\left[I(x > u)I(y > v)\right]dvdu = \int_{d_x}^{l_x}\int_{d_y}^{l_y} S(u,v)dvdu. \quad \blacksquare$$

7.4 Proof of Proposition 3.2:

We again use mathematical induction. For $i = j = 1$, the statement is true by proposition 3.1. Assume that it is true for i, j, then

$$\begin{aligned} R_{i+1,j}(l_x, l_y) &= \int_{l_x}^{\infty} R_{i,j}(u, l_y) du \\ &= \int_{l_x}^{\infty} \frac{1}{i!j!} \mathbb{E}[(X-u)_+^i (Y-l_y)_+^j] du \\ &= \frac{1}{i!j!} \mathbb{E}\left[(Y-l_y)_+^j \int_{l_x}^{\infty} (X-u)_+^i du\right] \\ &= \frac{1}{(i+1)!j!} \mathbb{E}\left[(X-l_x)_+^{i+1} (Y-l_y)_+^j\right]. \end{aligned}$$

The derivation for $R_{i,j+1}(l_x, l_y)$ is symmetric. ■