Pricing Catastrophe Excess of Loss Reinsurance using Power Curves and the Generalized Logarithmic Mean

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Abstract

In this paper we advocate the use of the Generalized Logarithmic Mean as midpoint of property catastrophe reinsurance layers when fitting rates on line with power curves. We show that the method is easy to implement and overcomes issues encountered when working with usual candidates for the midpoint like the arithmetic, geometric and logarithmic means. We also show how to deal with paid reinstatements in a simplified framework. We also introduce a new midpoint that is consistent with a negative exponential fit of the rates on line.

Keywords

Midpoint, Arithmetic Mean, Geometric Mean, Logarithmic Mean, Generalized Logarithmic Mean, Pareto Distribution, Rate on Line, Reinstatement, Survival Function, Property Catastrophe Excess of Loss Reinsurance, Power Curve, Negative Exponential Curve.

1 Introduction

In a recent paper, Morel (2013) discussed the use of power curves and midpoints of the reinsurance layers to price catastrophe excess of loss contracts. Morel (2013) pinpointed some flaws when using power curves and the midpoint of the reinsurance layers being the arithmetic mean or the geometric mean. Morel (2013) advocates to replace the power curve by spline functions to solve the issues. In the present paper we will highlight other important flaws in the power curve method. We will suggest a simpler procedure using power curves with their natural midpoint being the Generalized Logarithmic Mean. The paper is organized as follows. Section 2 introduces the problem and the numerical example which will be worked throughout the paper. Section 3 sketches how to deal with paid reinstatements. Section 4 introduces the European Pareto Distribution. Section 5 shows that the power curves method implicitly assumes that the prices behave according to a European Pareto distribution. Section 6 introduces some possible midpoints and shows related issues. Section 7 explains why we will not follow the spline functions route introduced in Morel (2013). Section 8 shows that the natural midpoint when using power curves is the Generalized Logarithmic Mean. Section 9 introduces an alternative method based on the negative exponential distribution and its associated midpoint. The numerical example is further analyzed in section 10. Section 11 concludes.

2 The ROL method

Property catastrophe reinsurance offers insurance companies protection against losses due to natural catastrophes. This type of reinsurance is purchased by all insurance companies writing property business because it provides a cost-effective capital relief. In other words the margin ceded to the reinsurers is smaller than the cost of holding capital and therefore it makes sense to purchase that type of reinsurance. We will use the following European catastrophe excess of loss program throughout the paper :

	Limit		Attachment Point	ROL	Premium	Reinstatements
Layer 1	90.000.000	xs	110.000.000	12.00%	10.800.000	2 at 100%
Layer 2	300.000.000	xs	200.000.000	4.50%	13.500.000	1 at 100%
Layer 3	300.000.000	xs	500.000.000	2.20%	6.600.000	1 at 100%
Layer 4	250.000.000	xs	800.000.000	1.20%	3.000.000	1 at 100%
Total program	940.000.000	xs	110.000.000	3.61%	33.900.000	

Table 2.1: Original Program

The rate on line (ROL) is simply the upfront premium divided by the limit of the layer.
In practice, the various layers of the program have a limited number (most of time one) of reinstatements and furthermore the reinstatements are usually payable at 100%. We will briefly discuss in Section 3 how to deal with this feature.
Reinsurers tend to use commercial models to price natural catastrophe perils. Further they all have their own pricing models in order to factor their administration and capital costs into

the commercial premium. Often reinsurance brokers and underwriters try to "fit" observed rates on line in order to extrapolate premiums to other layers and/or in order to predict premiums based on the evolution of the exposure and their anticipation of the pricing trends.

The aim of this paper is to justify a method commonly used by reinsurance actuaries that is based on power curves. We will show that when using a well chosen midpoint for the reinsurance layers, the method delivers consistent results.

3 Dealing with Paid Reinstatements

Most of the time, the reinsurance layers will have their yearly liability being limited to two or more times the limit (denoted by C) of the layer. In principle, the cedant will purchase a sufficient number of limits, meaning that this feature will have a marginal influence on the price which can be assumed to be theoretically valid for an unlimited number of reinstatements, even though reinsurers would not provide an unlimited yearly capacity for property cat business.

Furthermore reinstating the limits generally is not free. An additional premium called the reinstatement premium must be paid. We will assume the most general case where the reinstatement premium is equal to 100% of the initial premium multiplied by the reisured loss divided by the limit (see formula (3.1) below). We say that the reinstatements are payable at 100% pro rata capita. Reinstating the limit is not an option. So a loss to the layer will lead to a reinstatement premium (if there remains limits to reinstate). A layer with paid reinstatements implies that the upfront cost of the layer is smaller than in the case where the reinstatement(s) are free. Working with paid reinstatements versus free reinstatements leads to a rebate on the initial reinsurance premium.

Let us define S the stochastic loss in the layer and $SLOL = \frac{S}{C}$ the stochastic loss on line. If ROL is the upfront rate on line, Walhin (2001) shows that the expected additional ROL due to the paid reinstatements is given by

$$ROL \times \sum_{i=1}^{k} \frac{c_i}{C} \mathbb{E}[\min(C, \max(0, S - (i-1)C))], \tag{3.1}$$

where k is the number of reinstatements and c_i is the price of the ith reinstatement.
Here we assume that all reinstatements are paid at 100% and that the number of reinstatements is large enough so that we can approximate $k \to \infty$. Then formula (3.1) simplifies into

$$ROL \times \frac{\mathbb{E}[S]}{C} = ROL \times LOL,$$

where the loss on line LOL is equal to the expected value of the stochastic loss on line.

Therefore the rebate that can be given for paid reinstatements against free reinstatements is equal to LOL. In other words, if the layer has one (or more) reinstatements at 100%, then the equivalent ROL with free reinstatements (denoted $FROL$) is approximated by $ROL \times (1 + LOL)$. The LOL is not readily available but if one makes an assumption about the loading charged by reinsurers, it is possible to deduce the LOL and therefore the corresponding upfront rate on line ($FROL$) when reinstatements are free. Let us assume that $FROL$ is obtained by adding to the LOL 5% of the standard deviation of the stochastic loss on line being approximated by $\sqrt{LOL \times (1 - LOL)}$, and loading by 100/90. These parameters denote very soft conditions. The equation to solve numerically is :

$$\frac{LOL + 5\%\sqrt{LOL \times (1 - LOL)}}{0.9} = ROL \times (1 + LOL) = FROL.$$

The table below provides LOL and $FROL$ for our numerical example.

	Limit		Attachment Point	ROL	LOL	FROL
Layer 1	90.000.000	xs	110.000.000	12.00%	10.40%	13.25%
Layer 2	300.000.000	xs	200.000.000	4.50%	3.29%	4.65%
Layer 3	300.000.000	xs	500.000.000	2.20%	1.42%	2.23%
Layer 4	250.000.000	xs	800.000.000	1.20%	0.68%	1.21%

Table 3.1: ROL, LOL and FROL

With the above table, the user has the choice between fitting

1. $FROL$, that is the equivalent rate on line with free reinstatements
2. LOL, that is the loss on line
3. ROL, that is the upfront rate on line (in the case where reinstatements are paid). In principle, this would not be recommended because this method ignores the fact that the rebate due to the paid reinstatements is embedded in the value of ROL. See table 10.6 and related comments in section 10 for more details.

Let us emphasize on the fact that the above calculations are oversimplified. In practice underwriters would use their modelled stochastic loss on line to compute precisely LOL and the expected additional reinstatement premiums. Furthermore the underwriter would apply the profitability model of the reinsurer to price the layers are arrive at $FROL$ or ROL. In this paper, we do as if we do not have these informations at disposal. Instead we want to make quick calculations to compare the pricing of various layers. That is what the rate on line method is aiming at.

4 The European Pareto Distribution and the Pure Reinsurance Premium

The European Pareto distribution dates back to Pareto (1895) who studied the distribution of the revenues in a given population. It has been advocated by Hagstroem (1925) for using in reinsurance. We will say that X follows a Pareto distribution ($X \sim Pa(A, \alpha)$) if the cumulative distribution function of X is given by

$$F(x) = \mathbb{P}[X \leq x] = 1 - \left(\frac{x}{A}\right)^{-\alpha} \quad , \quad x > A.$$

The Pareto distribution is used by reinsurance actuaries because of its many nice mathematical properties (see Philbrick (1985) or Walhin (2003) for a discussion). Amongst others, the Pareto distribution is a particular case of the GPD (Generalized Pareto Distribution) introduced by Pickands (1975). The GPD can be shown to be the limiting case for the distribution of excesses above large thresholds, which is exactly the problem reinsurance actuaries look at.

Let us assume a layer C xs P. We will use the notation $c = \frac{C}{P}$.
Let N_A be the number of losses in excess of A with $A \leq P$.
Let $X_1, X_2, \ldots$ be the large losses. We will assume that they are mutually independent and identically distributed according to a Pareto distribution. The yearly liability of the reinsurer (with an unlimited number of reinstatements) writes :

$$S = \min(cP, \max(0, X_1 - P)) + \cdots + \min(cP, \max(0, X_{N_A} - P)).$$

The pure reinsurance premium ($PRP(P, C)$) for a layer C xs P is given by

$$PRP(P, C) = \mathbb{E}N_A \mathbb{E} \min(C, \max(0, X - P)) = \begin{cases} \mathbb{E}N_A \frac{P^{1-\alpha}A^{\alpha}}{1-\alpha}((1+c)^{1-\alpha} - 1) & \text{if } \alpha \neq 1 \\ \mathbb{E}N_A A \ln(1+c) & \text{if } \alpha = 1. \end{cases}$$

The loss on line ($LOL(P, C)$) is therefore :

$$LOL(P, C) = \frac{PRP(P, C)}{C} = \begin{cases} \mathbb{E}N_A \frac{A^{\alpha}}{P^{\alpha}} \frac{((1+c)^{1-\alpha}-1)}{(1-\alpha)c} & \text{if } \alpha \neq 1 \\ \mathbb{E}N_A \frac{A}{P} \frac{\ln(1+c)}{c} & \text{if } \alpha = 1. \end{cases}$$

It is also worth noting that the pure reinsurance premium of an unlimited layer ∞ xs P is given by

$$PRP(P, \infty) = \frac{A^{\alpha}}{\alpha - 1} P^{1-\alpha} \text{ if } \alpha > 1, \tag{4.1}$$

and exists only if $\alpha > 1$.

5 The Midpoint Method for Fitting ROL's

As explained in Morel (2013), a possible solution to the problem introduced in section 2 is to fit a power curve through midpoints of the original program layers. Morel (2013) claims that there is no literature on the subject. However Verlaak et al (2005) already justified the use of power curves in a Pareto framework.

Let us assume that the ROL is based on a frequency distribution with mean λ and a severity distribution X with survival function $\mathbb{P}[X > x] = S(x)\,,\, x \geq 0$.
We have that

$$ROL(P, C) = \frac{\lambda}{C} \int_{P}^{P+C} S(x)dx.$$

As $S(x)$ is a decreasing function we immediately find that

$$\lambda S(P + C) \leq ROL(P, C) \leq \lambda S(P).$$

Let us note $\lambda_x = \lambda S(x)$. We then have

$$\lambda_{P+C} \leq ROL(P, C) \leq \lambda_P.$$

Therefore there exists a point $x = MP(P, C)$ such that

$$\lambda_{P+C} \leq \lambda_{MP(P,C)} \approx ROL(P, C) \leq \lambda_P.$$

If $\lambda_{MP(P,C)}$ can easily be calculated, then an approximation for the $ROL(P, C)$ is provided based on a certain midpoint $MP(P, C)$ of the layer C xs P.
Fitting a power curve of the type

$$ROL(P_i, C_i) \approx \lambda_{MP(P_i,C_i)} = a[MP(P_i, C_i)]^{-b}$$

seems natural and will lead to estimating the parameters by linear regression. In fact assuming a power curve corresponds to the case where the severity of the process is Pareto distributed. Indeed, we have

$$\lambda_{MP(P,C)} = \lambda_A \left(\frac{MP(P, C)}{A} \right)^{-\alpha}, \tag{5.1}$$

where the midpoints are normalized by the parameter A and the λ parameter depends on the chosen value of A.
A can be arbitrarily choosen below the smallest attachment point of the programme (otherwise not all the losses hitting the programme would be modelled). Further if we assume $B < A$, we have :

$$\begin{aligned}
\lambda_{MP(P,C)} &= \lambda_A \left(\frac{x}{A}\right)^{-\alpha} \\
&= \lambda_B \left(\frac{A}{B}\right)^{-\alpha} \left(\frac{x}{A}\right)^{-\alpha} \\
&= \lambda_B \left(\frac{x}{B}\right)^{-\alpha}
\end{aligned}$$

showing that opting for any threshold lower than the lowest attachment point of the reinsurance programme will lead to the same α.

As briefly explained in Verlaak et al. (2005), formula (5.1) justifies the use of a power function to fit the observed ROL's. Its parameters are easily adjusted by linear regression after log transform. Assume that we have observed n layers (P_i, C_i) with ROL_i, $i = 1, \ldots, n$. We have

$$ROL_i = ROL(P_i, C_i) = \lambda_A y_i^{-\alpha} \quad \text{with} \quad y_i = \frac{MP(P_i, C_i)}{A} \quad , \quad i = 1, \ldots, n.$$

By taking the ln on both sides of the equality we have

$$\ln(ROL_i) = \ln(\lambda_A) - \alpha \ln(y_i) \quad , \quad i = 1, \ldots, n,$$

and the parameters (λ_A, α) can easily be estimated by linear regression. This method therefore does not require any numerical procedure, which is comfortable for reinsurance brokers and underwriters in order to produce quick calculations.

It remains to choose the midpoint, which we will do in section 6.

6 Candidates for the Midpoint and Various Issues

Morel (2013) used two midpoints : the arithmetic mean and the geometric mean. The geometric mean has the nice feature that it corresponds exactly to the case $\alpha = 2$ in the Pareto setting. A third interesting case could be the logarithmic mean, corresponding exactly to the case $\alpha = 1$ in the Pareto setting. We have :

$$\begin{aligned}
ARI(P, C) &= \frac{P + (P + C)}{2} = P\left(\frac{2 + c}{2}\right), \\
GEO(P, C) &= \sqrt{P(P + C)} = P\sqrt{1 + c}, \\
LOG(P, C) &= \frac{(P + C) - P}{\ln(P + C) - \ln(P)} = P\frac{c}{\ln(1 + c)}.
\end{aligned}$$

With the above midpoints, the approximated ROL (we write ROL but the reasoning is valid for LOL and $FROL$) becomes :

$$\begin{aligned}
ROL^{ARI}(P, C) &= \lambda_A \frac{P^{-\alpha}}{A^{-\alpha}}(1 + \frac{c}{2})^{-\alpha}, \\
ROL^{GEO}(P, C) &= \lambda_A \frac{P^{-\alpha}}{A^{-\alpha}}(1 + c)^{-\alpha/2}, \\
ROL^{LOG}(P, C) &= \lambda_A \frac{P^{-\alpha}}{A^{-\alpha}}(\frac{\ln(1 + c)}{c})^{\alpha},
\end{aligned}$$

and we immediately obtain the reinsurance premium as

$$
\begin{aligned}
RP^{ARI}(P,C) &= \lambda_A \frac{P^{1-\alpha}}{A^{-\alpha}} c(1+\frac{c}{2})^{-\alpha}, \\
RP^{GEO}(P,C) &= \lambda_A \frac{P^{1-\alpha}}{A^{-\alpha}} c(1+c)^{-\alpha/2}, \\
RP^{LOG}(P,C) &= \lambda_A \frac{P^{1-\alpha}}{A^{-\alpha}} c \left(\frac{\ln(1+c)}{c}\right)^{\alpha}.
\end{aligned}
$$

The power curves and associated fits for these three midpoints are given in tables 6.1 and 6.2. A has been chosen equal to 50.000.000.

Layer	ROL	ARI	ROL^{ARI}	GEO	ROL^{GEO}	LOG	ROL^{LOG}
1	12%	3.1	12.33%	2.97	11.92%	3.01	12.07%
2	4.50%	7	4.41%	6.32	4.70%	6.55	4.60%
3	2.20%	13	2.02%	12.65	2.00%	12.77	2.01%
4	1.20%	18.5	1.30%	18.33	1.27%	18.39	1.28%
Total program	3.61%		3.58%		3.62%		3.60%
Error			-0.76%		0.34%		-0.04%

Table 6.1: Original Program : adjusted ROL's with linear regression

	α	λ
ARI	1.2613	0.5138
GEO	1.2299	0.4542
LOG	1.2410	0.4738

Table 6.2: Original Program : fitted parameters through linear regression

We observe that the fits based on the three proposed midpoints are of relatively good quality. The best fit is obtained with the logarithmic mean, which is not surprising as the fitted α is around 1.25, ie not far from 1 (which corresponds to the logarithmic mean case). Let us know discuss various issues linked to the arbitrary choice a one of these midpoints.

ISSUE 1 Morel (2013) claims that the quality of the fit is not excellent and in particular that the total premium is not matched. We believe that this is unavoidable when using parcimonious mathematical models. One could argue that the fit could be enhanced by using weights when fitting the parameters of the linear regression. Natural weights are the premiums. Tables 6.3 and 6.4 provide the fits through weighted linear regression :

Layer	ROL	Weight	ARI	ROL^{ARI}	GEO	ROL^{GEO}	LOG	ROL^{LOG}
1	12%	10.800.000	3.10	12.15%	2.97	11.77%	3.01	11.90%
2	4.50%	13.500.000	7	4.46%	6.32	4.69%	6.55	4.61%
3	2.20%	6.600.000	13	2.08%	12.65	2.02%	12.77	2.04%
4	1.20%	3.000.000	18.50	1.35%	18.33	1.29%	18.69	1.31%
Total program	3.61%			3.61%		3.61%		3.61%
Error				0.10%		0.13%		0.10%

Table 6.3: Original Program : adjusted ROL's with weighted linear regression

	α	λ
ARI	1.2310	0.4894
GEO	1.2146	0.4407
LOG	1.2209	0.4573

Table 6.4: Original Program : fitted parameters through weighted linear regression

There remains a difference between the observed total premium and the fitted total premium. The user could eg correct the λ parameter to force the total approximated ROL to match with the total observed ROL.

ISSUE 2 Morel (2013) also claimed that different layers may have the same ROL. We do not believe that this is an issue. The theory perfectly allows for various layers to have the same ROL. See the table 10.4 in the numerical illustration of section 10.

ISSUE 3 Morel (2013) also claimed that due to the unboundness of the power curve, a layer attaching at an infinitely small level would have an infinite premium. That is in fact not true in all cases. In particular we have :

$$\begin{aligned}
\lim_{P\to 0} RP^{ARI}(P,C) &\rightarrow \lambda_A (2A)^{\alpha} C^{1-\alpha}, \\
\lim_{P\to 0} RP^{GEO}(P,C) &\rightarrow \infty, \\
\lim_{P\to 0} RP^{LOG}(P,C) &\rightarrow \infty.
\end{aligned}$$

So we observe that the limit does exist if the midpoint is the arithmetic mean. Anyway we believe that extrapolating to layers with infinitely small deductibles does not make sense and does not need to be captured by the model. It is indeed most likely that the exposure at such low levels cannot be be extrapolated from the exposure at higher levels. Attritional losses will require different modeling than large losses.

ISSUE 4 We also have that the price for adjacent layers is not additive, which is obviously a nonsense :

$$
\begin{aligned}
RP^{ARI}(P_i, C_i) + RP^{ARI}(P_i + C_i, C_j) &\neq RP^{ARI}(P_i, C_i + C_j), && (6.1)\\
RP^{GEO}(P_i, C_i) + RP^{GEO}(P_i + C_i, C_j) &\neq RP^{GEO}(P_i, C_i + C_j) \quad , \quad \alpha \neq 2, && (6.2)\\
RP^{LOG}(P_i, C_i) + RP^{LOG}(P_i + C_i, C_j) &\neq RP^{LOG}(P_i, C_i + C_j) \quad , \quad \alpha \neq 1. && (6.3)
\end{aligned}
$$

ISSUE 5 Other issues not mentioned in Morel (2013) are

$$
\begin{aligned}
\lim_{C\to\infty} RP^{GEO}(P, C) &\to \infty \quad , \quad \alpha < 2 && (6.4)\\
\lim_{C\to\infty} RP^{ARI}(P, C) &\to \infty \quad , \quad \alpha < 1 && (6.5)\\
\lim_{C\to\infty} RP^{LOG}(P, C) &\to \infty \quad , \quad \alpha \leq 1 && (6.6)\\
\lim_{C\to\infty} RP^{GEO}(P, C) &\to 0 \quad , \quad \alpha > 2 && (6.7)\\
\lim_{C\to\infty} RP^{ARI}(P, C) &\to 0 \quad , \quad \alpha > 1 && (6.8)\\
\lim_{C\to\infty} RP^{LOG}(P, C) &\to 0 \quad , \quad \alpha > 1 && (6.9)\\
\lim_{C\to\infty} RP^{GEO}(P, C) &\to \lambda_A \frac{A^2}{P} \quad , \quad \alpha = 2 && (6.10)\\
\lim_{C\to\infty} RP^{ARI}(P, C) &\to 2\lambda_A A \quad , \quad \alpha = 1. && (6.11)
\end{aligned}
$$

Limit (6.4) makes no sense for the cases $1 < \alpha < 2$. Indeed premiums for unlimited reinsurance layers remain finite when (loaded) claims are Pareto distributed with $\alpha > 1$ (see formula (4.1)).

Limits (6.5) and (6.6) make sense as the α parameter is smaller than 1 and the expectation of a Pareto random variable does not exist in this case.

Limits (6.7), (6.8) and (6.9) make no sense. Unlimited reinsurance layers must lead to non zero premiums.

Finite and non zero limits are only obtained for the very particular cases in (6.10) and (6.11), confirming again the danger of the method when dealing with layers having a huge limit.

7 The Spline Solution

Morel (2013) suggested to overcome most of the issues by instead of working with power curves and midpoints, integrating spline functions over the various layers. In fact, Morel (2013) implicitly used the formula :

$$RP(P, C) = \lambda_A \int_P^{P+C} S(x)dx,$$

where $S(x)$ is the survival function of the underlying claims or premium process. Morel (2013) suggested to fit spline functions. With this Morel (2013) overcomes issues 1 to 5. However that happens at the cost of introducing heavy assumptions about maximum ROL in the bottom of the programme as well as minimum ROL at the top of the programme. These concepts can easily be dealt by the user outside the model by using expert judgment. More importantly, the survival function is a decreasing function and the spline functions are not everywhere decreasing, leading to possible negative prices for certain layers. Eventually all this is based on a model which is heavily overparametrized. Morel (2013) used 19 parameters to fit 5 layers.

We propose in the next section a method that will overcome most issues and remain parcimonious in terms of number of parameters.

8 The Generalized Logarithmic Mean of order r

A two-variable continuous function $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is called a mean on $\mathbb{R}_+$ if $min(x,y) \leq f(x,y) \leq \max(x,y)$, $x, y \in \mathbb{R}_+$ holds.

A way to build a mean is to resort to the Cauchy Mean Value Theorem (see Cauchy (1882)). Let the functions $f(z)$ and $g(z)$ be continuous on an interval $[x, y]$, differentiable on (x, y), and $g'(z) \neq 0$ for all $z \in (x, y)$. Then there exists a point $z = \xi$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)}.$$

Let us choose $f(x) = x^r$ and $g(x) = x$. We have

$$\frac{x^r - y^r}{x - y} = r\xi^{r-1}.$$

Solving in ξ, we find

$$\xi = \left[\frac{x^r - y^r}{r(x - y)}\right]^{\frac{1}{r-1}}.$$

ξ is called the generalized logarithmic mean of x and y.

As we are interested in midpoints of layers $[P, P+C]$, which we denote by $MP(P, C)$, we will adopt the notation $MP(x, y - x)$ with $y \geq x$. In this context, we can write more precisely the generalized logarithmic mean as

$$L_r(x, y - x) = \begin{cases} \sqrt[r-1]{\frac{x^r - y^r}{r(x-y)}} & , \quad r \neq 0, r \neq 1, x \neq y \\ IDENTRIC(x, y - x) = e^{-1}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}} & , \quad r = 0, x \neq y \\ LOG(x, y - x) = \frac{x-y}{\ln(x) - \ln(y)} & , \quad r = 1, x \neq y \\ x & , \quad x = y. \end{cases}$$

The generalized logarithmic mean of order r has been introduced by Galvani (1927). It is sometimes called extended logarithmic mean and often presented as a particular case of the Stolarsky mean with two parameters introduced by Stolarsky (1975).

Stolarsky (1975) showed that when $x \neq y$, $L_r(x, y-x)$ is stricly increasing with r. We have the following particular cases :

$$
\begin{aligned}
\lim_{r\to-\infty} L_r(P,C) &= P, \\
L_{-2}(P,C) &= \left(\frac{(GEO(P,C))^4}{ARI(P,C)}\right)^{1/3}, \\
L_{-1}(P,C) &= GEO(P,C), \\
L_0(P,C) &= LOG(P,C), \\
L_{1/2}(P,C) &= QUAD(P,C) = \frac{1}{2}(ARI(P,C) + GEO(P,C)), \\
L_1(P,C) &= IDENTRIC(P,C), \\
L_2(P,C) &= ARI(P,C), \\
\lim_{r\to\infty} L_r(P,C) &= P + C.
\end{aligned}
$$

The generalized logarithmic mean has been extensively researched by mathematicians to prove various inequalities. See eg Wang et al. (2012) and Qiu et al. (2011).
The generalized logarithmic mean also has applications in convex function theory, in economics and in physics. See eg Guo and Ki (2001), Pittenger (1985), Kahlig and Matkowski (1996) or Pòlya and Szergö (1951). In this paper we will make a link with excess of loss layers priced with a European Pareto distribution.

Now let us make the following change of variable $\alpha = 1 - r$.
The Generalized Logarithmic Mean of order $1-\alpha$ is the midpoint which provides an exact formula for the ROL when $\alpha > 0$:

$$
ROL(P,C) = \lambda_A \left(\frac{L_{1-\alpha}(P,C)}{A}\right)^{-\alpha} = \begin{cases} \lambda_A \frac{P^{-\alpha}A^{\alpha}}{c(1-\alpha)}((1+c)^{1-\alpha} - 1) & \text{if } \alpha \neq 1 \\ \lambda_A \frac{A}{P}\frac{\ln(1+c)}{c} & \text{if } \alpha = 1. \end{cases}
$$

It therefore becomes logical to make the fit with midpoints being calculated according to the Generalized Logarithmic Mean of order $1-\alpha$ with the estimated α parameter. A very limited number of iterations will be required to obtain the fit based on the Generalized Logarithmic Mean as exemplified in table 8.1.

Iteration	α in $L_{1-\alpha}$	α Power curve fit	λ_A Power curve fit
1	2	1.21	0.44
2	1.21	1.2196	0.4537
3	1.2196	1.219619	0.453622
4	1.219619	1.21961926	0.45362245
5	1.21961926	1.2196192651	0.4536224457
6	1.2196192651	1.2196192650	0.4536224457

Table 8.1: Iterations to obtain the Power Curve fit with the corresponding $L_{1-\alpha}$

Table 8.2 provides the adjusted ROL with the generalized logarithmic mean as midpoint.

	ROL	Weight	$L_{1-\alpha}$	$ROL^{L_{1-\alpha}}$
Layer 1	12.00%	10.800.000	3.00	11.87%
Layer 2	4.50%	13.500.000	6.50	4.63%
Layer 3	2.20%	6.600.000	12.74	2.04%
Layer 4	1.20%	3.000.000	18.37	1.30%
Total program	3.61%		3.61%	
Error			0.11%	

Table 8.2: Original Program : adjusted ROL's with weighted linear regression

When comparing with table 6.3, we can't claim that the fit is visually better than with the other midpoints. But that is not the goal of using the generalized logarithmic mean. We will see in section 10 that the issues encountered in section 6 disapear when using the generalized logarithmic mean, which is why we advocate the generalized logarithmic mean in a parcimonious mathematical model.

It is also worth noting that Bobtcheff (2003) used the generalized logarithmic mean to fit property cat market curves by using non linear regression. In her master thesis, Bobtcheff (2003) used the rather intuitive terminology "Pareto Layer Mean" to define the midpoint of the layer.

9 Negative Exponential Setting

In section 5, we have assumed that the ROL's behave according to a power curve. Another straightforward parametric assumption would be to assume a negative exponential behaviour :

$$ROL_i = a\exp(-bMP(P_i, C_i)).$$

This case exactly corresponds to a severity being distributed according to a negative exponential distribution with survival function :

$$S(x) = \exp(-x/\theta)\,,\; x > 0.$$

Let us now find the midpoint that is matching the exact value of the ROL in the negative exponential setting. The exact ROL is given by

$$ROL(P,C) = \frac{\lambda}{C}\int_P^{P+C} \exp(-x/\theta) = \frac{\lambda\theta}{C}\left[\exp(-P/\theta) - \exp(-(P+C)/\theta)\right].$$

The approximated value of the ROL is given by

$$ROL(P,C) \approx \lambda S(MP(P,C)) = \lambda \exp(-MP(P,C)/\theta).$$

The midpoint (let us call it $EXP(P,C)$) matching the exact value of the ROL will be the solution of the equation

$$\frac{\lambda\theta}{C}\left[\exp(-P/\theta) - \exp(-(P+C)/\theta)\right] = \lambda \exp(-MP(P,C)\theta)$$

and we find

$$EXP(P,C) = P - \theta \ln\left[\frac{\theta}{C}\left(1 - \exp(-C/\theta)\right)\right].$$

This midpoint corresponds to the case $f(x) = \exp(-x/\theta)$ and $g(x) = x$ in the Cauchy Mean Value Theorem.

Table 9.1 shows the iterations to find the parameters.

Iteration	θ MP	θ Power curve fit
1	100.000.000	321.753.981
2	321.753.981	325.935.165
3	325.935.165	325.963.443
4	325.963.443	325.963.632
5	325.963.632	325.963.634
6	325.963.634	325.963.634

Table 9.1: Iterations to obtain the Negative Exponential Curve fit with the corresponding EXP Midpoint

Table 9.2 provides the adjusted ROL with the EXP midpoint.

Layer	ROL	Weight	EXP	ROL^{EXP}
Layer 1	12.00%	10.080.000	153.965.265	9.80%
Layer 2	4.50%	13.500.000	338.575.780	5.56%
Layer 3	2.20%	6.600.000	638.575.780	2.22%
Layer 4	1.20%	3.000.000	917.049.667	0.94%
Total program	3.61%		3.67%	
Error			1.81%	

Table 9.2: Original Program : adjusted ROL's with weighted linear regression

We can visually see that the fit is of lower quality than the power curve fit.

10 Numerical Application Continued

In this section we will obtain the adjustment for $FROL$ as we believe it is more appropriate to work on fixed premiums.
Tables 10.1 and 10.2 provides the adjusted $FROL$'s with the generalized logarithmic, arithmetic and geometric means. A has been taken equal to 50.000.000.

Layer	FROL	Weight	$L_{1-\alpha}$	$FROL^{L_{1-\alpha}}$	GEO	$FROL^{GEO}$	ARI	$FROL^{ARI}$
1	13.25%	11.922.852	3.00	12.99%	2.97	12.88%	3.10	13.31%
2	4.65%	13.944.321	6.48	4.85%	6.32	4.92%	7.00	4.66%
3	2.23%	6.693.532	12.73	2.05%	12.65	2.04%	13.00	2.10%
4	1.21%	3.020.325	18.37	1.29%	18.33	1.27%	18.50	1.33%
Total program	3.79%			3.79%		3.79%		3.79%
Error				0.12%		0.16%		0.08%

Table 10.1: European Program : adjusted FROL's with weighted linear regression

	α	λ
$L_{1-\alpha}$	1.2759	0.5273
GEO	1.2712	0.5131
ARI	1.2874	0.5711

Table 10.2: European Program : fitted parameters through weighted linear regression

Assume that we have the following information for the next reinsurance renewal :

a) the tariff will drop by 5%,
b) the exposure will drop by 10%.

We will now extrapolate the price for various layers at the next renewal. We know that the exposure will drop by 10%. In order to take this into account, we will reduce the parameter A by 10%. This is easily justified by the fact that the parameter A is a scale parameter for the Pareto distribution. We find :

Layer	Limit	xs	Attachment Point	$RP^{L_{1-\alpha}}$	RP^{GEO}	RP^{ARI}
1	105.000.000	xs	95.000.000	12.994.383	12.981.505	13.005.383
2	250.000.000	xs	200.000.000	11.425.396	11.500.579	11.198.546
3	250.000.000	xs	450.000.000	5.229.471	5.186.779	5.372.198
4	250.000.000	xs	700.000.000	3.259.400	3.225.753	3.375.189
Sum 1 to 4	855.000.000	xs	95.000.000	32.908.649	32.894.616	32.951.316
5	855.000.000	xs	95.000.000	32.908.649	39.262.692	20.783.190
6	2.000.000.000	xs	95.000.000	40.177.173	55.556.225	18.753.540
7	10^{15}	xs	3.000.000.000	26.571.265	1.759.725.129	250.162

Table 10.3: New programme : various layers

We can make the following observations :

1. For the new programme in four layers, the sum of the price of the four layers is equal to the price of the equivalent programme in one layer with the generalized logarithmic mean. However it is not the case with the arithmetic and geometric means, as announced in 6.1 and 6.3.

2. The price of layer 6 is inconsistent with the arithmetic mean as it is lower than the price for a lower limit with the same attachment point.

3. Layer 7 illustrates equations 6.4 and 6.8. For the geometric mean, the premium tends to infinity, which is unexpected with an α parameter being larger than 1. On the other hand, the premium for the arithmetic mean case converges to 0 which is a non sense.

The above demonstrates again that we should work with the generalized logarithmic mean. We will concentrate on this case for the rest of the numerical application.

Let us also show that two different layers may have the same midpoint, and the same ROL (here $FROL$) :

Layer	Limit	xs	Attachment Point	$L_{1-\alpha}$	$FROL^{L_{1-\alpha}}$
8	15.000.000	xs	95.000.000	2.27	18.50%
9	25.818.000	xs	90.000.000	2.27	18.50%

Table 10.4: New programme : layers with the same MP and FROL

And obviously there are as many as you like layers with the same midpoint and ROL.

We still have to find the ROL for the various layers, that is applying the paid reinstatements as well as the known 5% price off. This is easily done by reducing $FROL$ by 5% and using the adapted equation to compute LOL and ROL :

$$(LOL + 5\%\sqrt{LOL * (1 - LOL)})\frac{0.95}{0.9} = ROL * (1 + LOL) = FROL.$$

Table 10.5 provides $FROL$, LOL and ROL.

	Limit		Attachment Point	$FROL$	LOL	ROL
Layer 1	105.000.000	xs	95.000.000	11.76%	9.66%	10.72%
Layer 2	250.000.000	xs	200.000.000	4.34%	3.23%	4.21%
Layer 3	250.000.000	xs	450.000.000	1.99%	1.31%	1.96%
Layer 4	250.000.000	xs	700.000.000	1.24%	0.74%	1.23%
Layer 5	855.000.000	xs	95.000.000	3.66%	2.66%	3.56%
Layer 6	15.000.000	xs	95.000.000	17.57%	14.87%	15.30%

Table 10.5: ROL, LOL and $FROL$ based on the $FROL$ adjustment

We will finalize the numerical application by showing the final results when $FROL$ is adjusted (which we did above), but also when LOL or ROL are adjusted. Table 10.6 provides the results.

	Limit		Attachment Point	ROL based on adjustment with $FROL$	LOL	ROL
Layer 1	105.000.000	xs	95.000.000	10.72%	10.80%	11.33%
Layer 2	250.000.000	xs	200.000.000	4.21%	4.15%	4.37%
Layer 3	250.000.000	xs	450.000.000	1.96%	1.96%	2.07%
Layer 4	250.000.000	xs	700.000.000	1.23%	1.26%	1.32%
Layer 5	855.000.000	xs	95.000.000	3.56%	3.65%	3.66%
Layer 6	15.000.000	xs	95.000.000	15.30%	15.61%	16.66%

Table 10.6: Final ROL based on $FROL$, LOL and ROL adjustments

We observe that the results with the $FROL$ and LOL adjustments are rather similar. However there are material deviations when we compare with the results obtained with ROL adjustment. The latter method ignores the paid reinstatements and is implying errors, in particular for layers with a high ROL, and therefore a larger impact of the paid reinstatement. Layers 1 and 6 are good examples. Thus, although the ROL method has the advantage that no assumption is needed to deduce LOL and / or $FROL$, we advocate to first deduce LOL and / or $FROL$ and work the adjustment on these variables as they are not impacted by the paid reinstatements.

11 Conclusion

In this paper we have analyzed how to make a quick analysis of the pricing of property cat excess of loss layers. The method is not too complex and can easily be implemented in a spreadsheet. If a power curve is chosen to fit the rates on line in function of a midpoint of the layers, we have argued that it is worth using its corresponding midpoint being the generalized logarithmic mean. Calculations are marginally more complex than with the usually used arithmetic, geometric or logarithmic means but will lead to consistent results. We have also shown how to take into account paid reinstatements in a simple way. The method can easily be used to compare the pricing of layers from one year to the other, but also to build benchmark curves when using data from various insurance companies.

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