Limited-Fluctuation Credibility with Uncertain Priors

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ABSTRACT

The classical notion of credibility is revisited, with an additional model explaining *uncertainty in the prior distribution.* This reflects practical situations where the prior distribution of parameters in the actuarial frequency-severity model is not fully deterministic, for example, when it is estimated from past experience. When both components of the compromise estimator are random, the limited-fluctuation credibility approach can result in three outcomes - there may be full credibility, partial credibility, or no credibility at all.

Three methods of evaluating credibility factors are proposed and compared, based on three different interpretations of limited fluctuation under the new general setting. Special methods are elaborated for dealing with *heterogeneous* risk groups.

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Keywords: Compromise estimator, Credibility, Hyper-parameters, Limited fluctuation, Prior distribution, Risk groups.

1. Introduction

1.1. Classical concept of credibility

The century-old concept of credibility presumes combining *real* data R (also called in actuarial literature as current observations, subject experience, or sample mean) with a *hypothetical* source H (external information, collective experience, relevant experience, prior mean) through a *compromise estimator*

$$C = ZR + (1 - Z)H, \tag{1.1}$$

where Z is the *credibility factor* (Longley-Cook, 1962; Hickman and Heacox, 1999; Bühlmann and Gisler, 2005; Klugman et al., 2012; Herzog, 2015). This approach is a standard method of valuation of insurance contracts in a typical situation when the actual history of an insured is insufficient to claim any reasonable estimation accuracy. In order to estimate the *pure premium*, a company needs a reliable predictor of the *total loss*, the amount it will have to pay in order to cover claims due to the occurrence of *insured events* such as traffic accidents or hospitalizations.

The total loss depends on the *frequency* and *severity* of insured events that are estimated from the past experience of the insured. When this experience is too short to produce a sufficient number of claims, the company supplements it with the *prior distribution* that is often based on the collective history of a risk group and other plausible sources. Similar situations often arise in business and finance, when new contracts are valuated in the presence of insufficient data or in a rapidly changing environment.

Credibility theory determines the minimal conditions under which a cohort is fully credible. *Full credibility* is assigned in this case, and the compromise estimator C is computed from the actual data only, thus receiving the credibility factor $Z = 1$. When full credibility is not possible, *partial credibility* is applied, with credibility factor $Z < 1$, and the compromise estimator is based on both the actual data and the prior information. Credibility factor Z plays a vital role in credibility estimation. It determines the portions of the compromise estimator C attributed to the real and the hypothetical data.

The *limited fluctuation credibility* approach is a standard way of determining allowed values of Z. It requires that in order to deserve a credibility factor Z, the real data must satisfy the following condition,

$$\boldsymbol{P}\left\{Z|R - \mathbf{E}(R)| > c\,\mathbf{E}(R)\right\} \leq \alpha, \tag{1.2}$$

for a given probability α and desired relative precision c.

Under the classical *frequency-severity model*, when the total loss $R = X = \sum_{i=1}^{N} Y_i$ consists of a Poisson(λ) number of individual losses Y_i, condition (1.2) is satisfied by a sufficiently large frequency of insured events λ. It follows from rather simple probability arguments that frequency

$$\lambda \geq \lambda_F = \left(\frac{z_{\alpha/2}}{c}\right)^2 (1 + \gamma^2) \tag{1.3}$$

suffices for the full credibility, where $\gamma = \sigma/\theta$ is a coefficient of variation of losses Y_i, $\theta = \mathbf{E}(Y_i)$ is the mean loss, σ is the loss standard deviation, and $z_{\alpha/2}$ is the upper $\alpha/2$-quantile of the standard normal distribution. It is assumed here that λ_F is sufficiently large to allow normal approximation of X (Herzog 2015, chap. 5; Klugman et al. 2012, chap. 17).

When $\lambda < \lambda_F$, the partial credibility condition (1.2) is satisfied with frequency

$$\lambda \geq \lambda_P = Z^2 \lambda_F. \tag{1.4}$$

Either full or partial credibility can *always* be assigned under these conditions because the hypothetical estimate H, equal to the prior mean μ with probability one, is assumed without any error or uncertainty. When the history of real data is short, (1.2) always holds with a zero or very small credibility factor. The lack of information contained in real data can always be compensated by "infinite information" contained in the prior distribution because it is assumed to be known completely, without any error or any uncertainty.

1.2. Two pointed limitations of the classical concept

Limitation 1: Error-Free Prior. The classical limited-fluctuation credibility approach accounts for uncertainty in the real data only, assuming no uncertainty in the hypothetical prior data and considering them as *fully credible*. This assumption has been pointed by Tindall and Mast (2003) as misleading and unrealistic in practical settings.

In actuarial practice, it is unrealistic to know prior parameters exactly, without any error or uncertainty. More typically, they are estimated from past experience, such as the frequency and severity of claims in a given risk group. As pointed in Tindall and Mast (2003), in practice, the amount of confidence in the prior mean ("current expectations" in the article) "drives the extent to which the actuary relies on credibility theory". If the prior mean was obtained as "purely a guess", then the actuary will rely more heavily on the current real data ("emerging experience").

In this article, we introduce *uncertainty* in the prior distribution parameters and express it in terms of a non-zero variance of the corresponding *hyper-prior*, the second-level prior, which is the prior distribution of the prior mean of the loss model. The limited-fluctuation condition is then revisited under this generalized model, resulting in corrected conditions for full and partial credibility. Several approaches are proposed; each of them provides a compromise estimator with a credibility factor determined by the uncertainty of real and hypothetical data. The standard criteria (1.3) and (1.4) for full and partial credibility appear as a special case of a fully credible prior, when the hyper-prior variance is zero.

A new third scenario then emerges, in addition to full and partial credibility. In the case when the degree of uncertainty is too high in *both* the real and the hypothetical data, even a partial credibility sometimes may not be possible. For example, this situation takes place when a totally new type of events is insured, with insufficient data as well as past experience.

Limitation 2: Agreement and Homogeneity. Tindall and Mast (2003) mention another limitation of the standard credibility practice. They emphasize a typical *heterogeneity* within the insured groups. Credibility depends on the degree of *agreement* between the actual data and the prior expectations. "The closer the data lean toward homogeneity and comparability with current expectations, the higher the credibility is likely to be". At the same time, Tindall and Mast (2003) notes that "experience, however, rarely matches contemporary expectations".

Actuarial Standards of Practice No. 25 (ASOP, December 2013) require that "in carrying out credibility procedures, the actuary should consider the homogeneity of both the subject experience and the relevant experience. Within each set of experience, there may be segments that are not representative of the experience set as a whole". Indeed, as long as variability exists within a risk group, it is possible for a given insured to have the distribution parameters different from the relevant risk group. For example, a driver with the same driving record, type of a vehicle, and family structure as the rest of the risk group may still be on the safer side or on the risky side of the group. As a result, $\mathbf{E}(X)$ for the given insured is no longer equal to the hyper-prior mean $\nu = \mathbf{E}(\mu)$. The real data and the hypothetical data may have different means.

We address this issue by accounting for heterogeneity, different distributions of real data

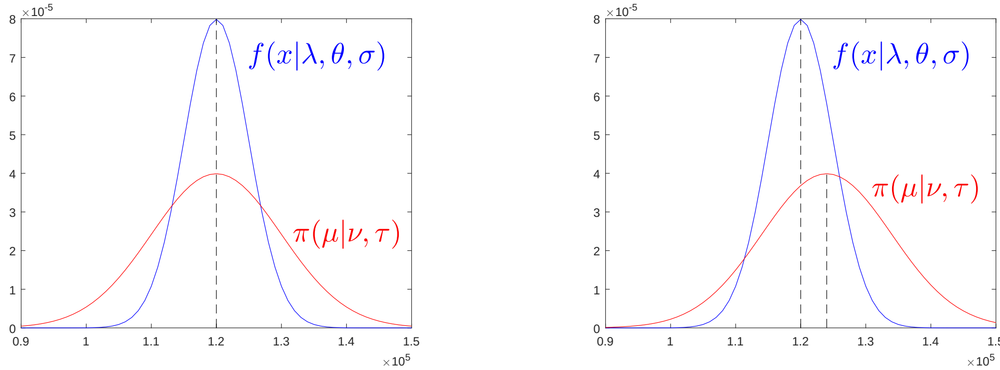

Figure 1: *Degree of agreement. On the left, the prior perfectly agrees with the distribution of losses, and* $\mathbf{E}(X) = \mathbf{E}(\mu) = \nu$*. On the right, there is some disagreement, as* $\mathbf{E}(X) \neq \nu$*.*

within each group. In Section 2, we assume "typical" or "random" members of risk groups, for whom the prior mean agrees with the actual expectation. An example of a perfect agreement is on Figure 1, left. Then, in Section 3, we consider insureds with some deviation from the prior mean, see Figure 1, right. Since the prior mean is no longer assumed to be fully known with no error, such deviations occur inevitably. The impact of a disagreement between the real data and the prior expectation is apparent. In the extreme case, when discordance between the current experience and expectations is too high, there may be no room even for a partial credibility. The last two sections of the paper contain an illustration of the proposed methods with specific scenarios, followed by a summary and conclusions.

2. Limited Fluctuation under Uncertainty in Hypothetical Data

Actuarial Standards of Practice No. 25 (December, 2013) state that "The actuary should use an appropriate credibility procedure when determining if the subject experience has full credibility or when blending the subject experience with the relevant experience. The procedure selected or developed may be different for different practice areas and applications".

We propose three limited-fluctuation methods that account for uncertainty of hypothetical data and then discuss the situations appropriate for each method.

To quantify this uncertainty, we assume that the prior mean μ of the total loss X is no longer deterministic. Rather, it has an approximately normal distribution with mean ν and variance τ^2. Sample mean and sample variance of losses within the risk group are often used to generate an estimate of μ, in which case τ^2 is inversely proportional to the group size. To summarize, we have

$$\begin{array}{ll} \text{Total loss:} & X = \sum_{i=1}^{N} Y_i \approx \text{Normal}, \\ \text{Frequency of claims:} & N \sim \text{Poisson}(\lambda), \\ \text{Severity parameters:} & \theta = \mathbf{E}(Y_i),\ \sigma^2 = \text{Var}(Y_i),\ \gamma = \sigma/\theta, \\ \text{Prior distribution:} & \mu \sim \text{Normal}(\nu, \tau^2). \end{array} \tag{2.1}$$

Under this general setting, the distribution of the total loss X has mean and variance

$$\begin{aligned} \mathbf{E}(X) &= \mathbf{E}\,\mathbf{E}\left\{\textstyle\sum_1^N Y_i \mid N\right\} = \mathbf{E}(N\theta) = \lambda\theta, \\ \text{Var}(X) &= \mathbf{E}\,\text{Var}\left\{\textstyle\sum_1^N Y_i \mid N\right\} + \text{Var}\,\mathbf{E}\left\{\textstyle\sum_1^N Y_i \mid N\right\} = \mathbf{E}(N\sigma^2) + \text{Var}(N\theta) = \lambda\sigma^2 + \lambda\theta^2. \end{aligned} \tag{2.2}$$

Parameters λ, θ, and σ are generally unknown. Thus, the chosen prior distribution and the H component of the compromise estimator (1.1) may *agree* or *disagree* with the actual distribution of X.

By *agreement*, we understand that the chosen prior distribution of μ has the mean ν that actually equals the mean of X. Notice that this does not always have to be the case. Actuaries have the real data, on one hand. On the other hand, they *choose* a prior distribution, or they obtain it from some database such as past experience of a relevant risk group.

In case of an *agreement*, $\mathbf{E}(X) = \lambda\theta = \mathbf{E}\,\mathbf{E}\left\{X \mid \mu\right\} = \mathbf{E}(\mu) = \nu$. This can be understood as an *unbiased* choice of the prior distribution. For example, suppose the prior distribution is generated by the relevant risk group, and the given insured is a typical member of the associated risk group, in which case the expected total loss is $\mathbf{E}(X) = \nu$. This is justified when an insured is selected from a group at random. Then the expected total loss agrees with the expectation across the risk group, as on Figure 1, left.

Consider a general situation when n years (or insured periods) of data are available, $X_1, \ldots, X_n$. The real component of the compromise estimator will then consist of the sample mean $R = \bar{X}$.

We now have two estimators of the expected total loss $\mathbf{E}(X)$ that complement each other - the sample mean $\bar{X}$ and the prior mean μ. Each of them can be used to estimate $\mathbf{E}(X)$, but it is most efficient to combine them, in accordance with the credibility and Bayesian principles. This results in the compromise estimator C.

In this respect, we introduce three limited-fluctuation credibility conditions that differ in their interpretation of precision while estimating the expected total loss. These conditions restrict deviations of estimates from the expected total loss in three different ways, forcing the sample mean and the prior mean to be close to $\mathbf{E}(X)$ separately, as in (2.3), or jointly, as in (2.5), or having the compromise estimator C close to $\mathbf{E}(X)$ in (2.6).

In this section, we assume that a given insured is a *typical member* of the associated risk group, in which case the expected total loss is $\mathbf{E}(X) = \nu$. For example, this is justified when an insured is selected from a group at random. Then the expected total loss agrees with the expectation across the risk group.

2.1. Method I: Fluctuations of Real and Hypothetical Data

The standard form of the compromise estimator $C = ZR + (1-Z)H$ contains *two credibility factors*. Factor Z shows credibility of real data whereas $(1-Z)$ corresponds to credibility of

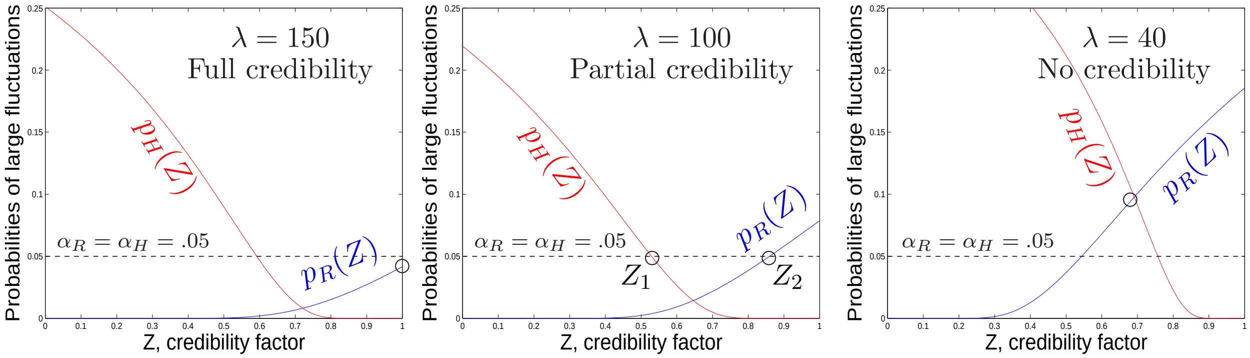

Figure 2: *Method I. Probabilities of large fluctuations for real and hypothetical data. Three situations: full credibility (left), partial credibility (middle), no credibility (right).*

the prior. Uncertainty in both R and H require symmetric conditions on coefficients Z and $(1-Z)$.

Thus, the classical condition (1.2) is now replaced by a combination of two conditions

$$\begin{cases} p_R &= \boldsymbol{P}\left\{Z|\bar{X} - \mathbf{E}(X)| > c\,\mathbf{E}(X)\right\} \leq \alpha_R \\ p_H &= \boldsymbol{P}\left\{(1-Z)|\mu - \mathbf{E}(X)| > k\,\mathbf{E}(X)\right\} \leq \alpha_H \end{cases} \tag{2.3}$$

for chosen probabilities α_R and α_H and relative precision factors c and k. The first inequality in (2.3) is the classical limited-fluctuation condition on the permissible deviation of real data X from its expected value. The second inequality represents a similar condition on the hypothetical data.

Let us look at the probabilities in (2.3) that are depicted in Figure 2, with $\alpha = \alpha_R = \alpha_H = 0.05$, $c = k = 0.05$, $\theta = 200$, $\sigma = 100$, $n = 10$, and different values of λ. Clearly, p_R increases with Z while p_H decreases. Regions where both curves appear below the threshold $\alpha_R = \alpha_H = 0.05$ contain the allowed credibility factors Z. For different frequency parameters λ, we notice three distinct cases.

1. Full credibility. When the expected frequency λ is sufficiently large, the solution set of (2.3) includes $Z = 1$. The insured is fully credible, and the pure premium can be estimated solely from the real data (Figure 2, left).

2. Partial credibility. With a lower λ, solutions of system (2.3) form an interval $[Z_1, Z_2]$ with $0 \le Z_1 \le Z_2 < 1$. Any credibility factor from this interval can be chosen. In practice, underwriters will often choose Z_2. The interval does not contain $Z = 1$, therefore, the insured is not fully credible (Figure 2, middle). Partial credibility applies in this case.

3. No credibility. Reducing λ even further, we enter the situation when there are no solutions of (2.3) in $0 \le Z \le 1$. It happens when variability of both the real and the prior data is so high that no Z-factor can satisfy both limited-fluctuation conditions simultaneously.

On Figure 2 (right), the intersection point of the two curves appears above $\alpha_R(= \alpha_H)$, and thus, conditions (2.3) are not satisfied together for any Z.

This case will never appear if the prior mean is (unrealistically) assumed to be exactly known. Such a 100% reliable estimate can always satisfy the limited-fluctuation condition. On the other hand, if both R and H are unreliable, then no compromise combination of them can "magically" become reliable!

2.1.1 Analytic Solution

To solve inequalities (2.3) analytically, we use the mean and variance of losses derived in (2.2) based on the assumptions (2.1). Then, inequalities in (2.3) are equivalent to

$$\frac{c\,\mathbf{E}(X)}{Z\,\mathrm{Std}(\bar{X})} \ge z_{\alpha_R/2} \quad \text{and} \quad \frac{k\,\mathbf{E}(X)}{(1-Z)\,\mathrm{Std}(\mu)} \ge z_{\alpha_H/2}.$$

Solving them for Z and using (2.2) and assumed agreement $\nu = \mathbf{E}(X)$, we obtain the solution in a form of two inequalities,

$$\begin{cases} Z & \le & \dfrac{c\,\mathbf{E}(X)}{z_{\alpha_R/2}\,\mathrm{Std}(\bar{X})} = \dfrac{c\lambda\theta}{z_{\alpha_R/2}\sqrt{\lambda(\theta^2+\sigma^2)/n}} = \dfrac{c\sqrt{\lambda n}}{z_{\alpha_R/2}\sqrt{1+\gamma^2}} \\ Z & \ge & 1 - \dfrac{k\,\mathbf{E}(X)}{z_{\alpha_H/2}\,\mathrm{Std}(\mu)} = 1 - \dfrac{k\nu}{z_{\alpha_H/2}\tau} \end{cases}$$

The interval of solutions

$$Z \in [Z_1, Z_2] = \begin{cases} \left[1 - \frac{k\nu}{z_{\alpha_H/2}\tau}, \frac{c\sqrt{\lambda n}}{z_{\alpha_R/2}\sqrt{1+\gamma^2}}\right] & \text{if} \quad 1 - \frac{k\nu}{z_{\alpha_H/2}\tau} \le \frac{c\sqrt{\lambda n}}{z_{\alpha_R/2}\sqrt{1+\gamma^2}} \\ \varnothing & \text{otherwise} \end{cases} \tag{2.4}$$

represents all credibility factors that can be assigned.

Similarly to the classical credibility theory, a higher expected frequency λ of insured events makes an insured more credible by increasing the upper bound of (2.4). As seen on Figure 2, increasing λ moves an insured from a "no credibility" category into a "partial credibility" and further, into a "full credibility". When the interval (2.4) contains $Z = 1$, full credibility can be assigned.

When the hypothetical mean μ is (unrealistically) considered fully credible, it corresponds to $\tau^2 = \text{Var}(\mu) = 0$. In this case, the lower bound of (2.4) becomes $-\infty$, converting (2.4) to the classical solutions for full and partial credibility (Herzog, 2015; Klugman et al., 2012).

We can also see that the solution (2.4) exists when the expected frequency of claims λ is high, reflecting informative real data, or when variance τ^2 is low, reflecting informative hypothetical data. When both sources R and H lack information, the interval is empty, and there no solution.

2.2. Method II: The Jointly Limited Fluctuation

Here we require that the real and prior estimates attain the desired relative precision *simultaneously* with a high probability $(1 - \alpha_2)$. In other words,

$$p_2 = \boldsymbol{P}\left\{Z|\bar{X} - \mathbf{E}(X)| > c\,\mathbf{E}(X) \;\cup\; (1-Z)|\mu - \mathbf{E}(X)| > k\,\mathbf{E}(X)\right\} \le \alpha_2. \tag{2.5}$$

Assuming independence of the real data and the prior parameters (such as independence of the given insured from all the other members of the risk group), condition (2.5) results in

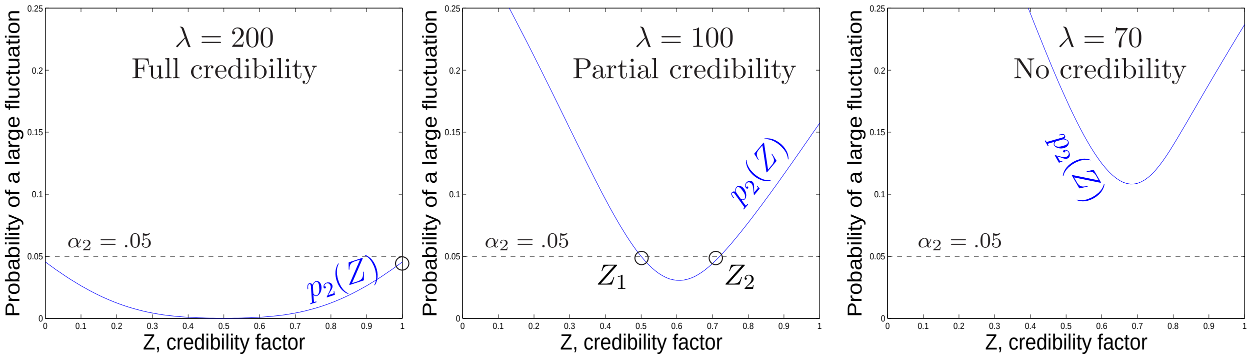

Figure 3: *Method II. Joint fluctuation probabilities. Full credibility (left), partial credibility (middle), no credibility (right).*

the inequality

$$p_2 = 1 - (1 - p_R)(1 - p_H) = 1 - \left\{1 - 2\Phi\left(-\frac{c\sqrt{\lambda n}}{Z\sqrt{1+\gamma^2}}\right)\right\}\left\{1 - 2\Phi\left(-\frac{k\nu}{(1-Z)\tau}\right)\right\} \leq \alpha_2,$$

where Φ denotes the standard normal distribution function. Notice that probability p_2 accounts for uncertainty in the present experience R as well as the hypothetical mean H. Hence, the probability in (2.5) is understood with respect to the *joint distribution* of $R = X$ and $H = \mu$.

Similarly to Method I, this may result in full, partial, or no credibility at all, and credibility increases with the expected frequency λ, as seen in Figure 3.

2.3. Method III: Fluctuation of the Compromise Estimator

Besides conditions on each component of the compromise estimator, it is natural to require a certain relative precision of the *whole estimator* C. After all, the compromise estimator $C = ZX + (1 - Z)\mu$ is the one to be used by the insurance company as the final estimator of the pure premium. Then, as the third approach, we require that

$$p_3 = \boldsymbol{P}\left\{|C - \mathbf{E}(X)| \geq c\,\mathbf{E}(X)\right\} = 2\Phi\left(-\frac{c\lambda\theta}{\sqrt{Z^2\lambda(\theta^2+\sigma^2)/n + (1-Z)^2\tau^2}}\right) \leq \alpha_3. \quad (2.6)$$

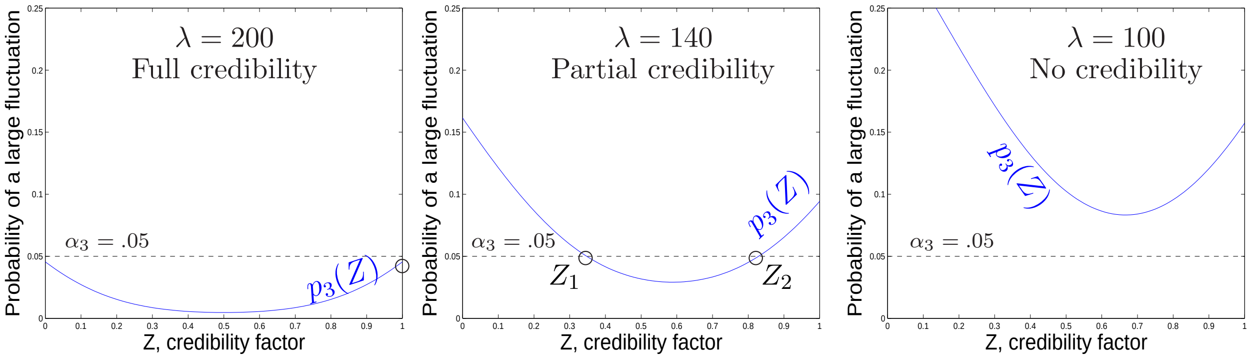

Figure 4: *Method III. Limited fluctuation probability of the compromise estimator. Full credibility (left), partial credibility (middle), no credibility (right).*

This condition can also result in full, partial, or no credibility. These three scenarios are seen in Figure 4.

The intervals of credibility factors Z solving (2.5) and (2.6) do not admit closed-form expressions. Both inequalities can be solved numerically.

3. Heterogeneity of Risk Groups

Kelliher et al. (2013) defines *heterogeneity risk* as one of the components of insurance risk that is related to "heterogeneity within risk groups used to set expectations, with variations in the profile of each risk group distorting experience". Due to heterogeneous groups, there are insured customers or cohorts whose expected loss differs from the prior mean, i.e., their actual experience is in a strong or weak disagreement with the prior distribution, as in our example in the end of Section 1 and in the right panel of Figure 1. This discordance affects obtained solutions of limited-fluctuation inequalities (2.3), (2.5), and (2.6) in the previous section, because now the prior mean is a *biased* estimator of the pure premium for the given insured.

Let $\delta = (\nu - \lambda\theta)/\tau$ be the relative difference between the hyper-prior (risk group) mean

ν and the expected loss $\mathbf{E}(X) = \lambda\theta$. In terms of δ, the large fluctuation probability of the prior mean in (2.3) becomes

$$p_H(\delta) = \boldsymbol{P}\left\{(1-Z)|\mu - \mathbf{E}(X)| > k\,\mathbf{E}(X)\right\} = \Phi\left(-\frac{k\lambda\theta}{(1-Z)\tau} + \delta\right) + \Phi\left(-\frac{k\lambda\theta}{(1-Z)\tau} - \delta\right).$$

Methods of Sections 2.1 and 2.2 make use of this probability, and thus, disagreement affects the credibility factors computed according to both of them. Method III of Section 2.3 requires bounding the probability

$$\begin{aligned} p_3 = p_3(\delta) &= \boldsymbol{P}\left\{|C - \mathbf{E}(X)| \geq c\,\mathbf{E}(X)\right\} \\ &= \Phi\left(\frac{-c\lambda\theta + \tau(1-Z)\delta}{\sqrt{Z^2\lambda(\theta^2+\sigma^2)/n + (1-Z)^2\tau^2}}\right) + \Phi\left(\frac{-c\lambda\theta - \tau(1-Z)\delta}{\sqrt{Z^2\lambda(\theta^2+\sigma^2)/n + (1-Z)^2\tau^2}}\right) \end{aligned}$$

that is now also dependent on δ, the relative deviation of the prior mean from the actual mean of real data.

The impact of δ on credibility estimates is seen in Figure 5 for all the three introduced methods. Larger differences between the expectations for an insured and the risk group cause the probability of a large fluctuation to increase. In this depicted case, a relative difference of $|\delta| \geq 2$ results in no available credibility estimator.

Figure 5 also shows that *the current experience appears more credible so long as it agrees more with the prior expectations.*

4. Illustration

To illustrate the proposed methods, here we consider several scenarios with different severity and frequency parameters θ, σ, and λ and hyper-parameters ν and τ of the prior distribution. In all scenarios, we suppose that $n = 3$ years of data are available for an insured. The tolerable error probabilities are set to be $\alpha_R = \alpha_H = 0.05$ for individual deviations in (2.3)

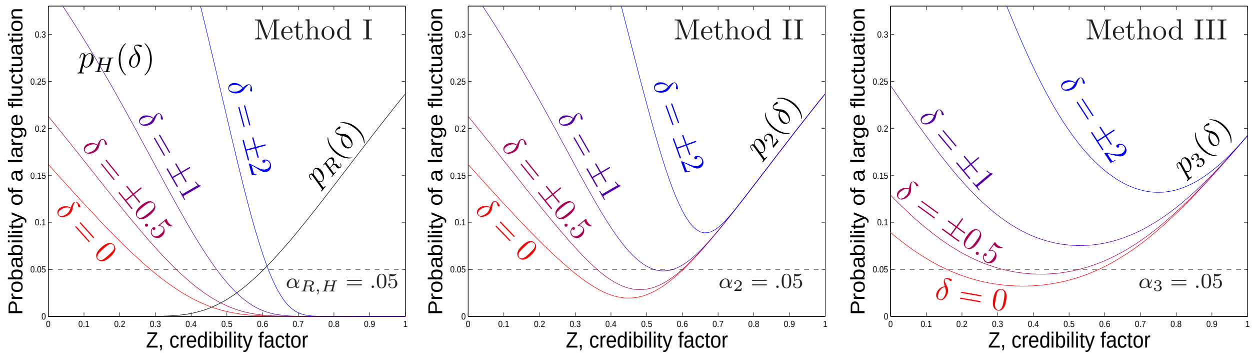

Figure 5: *Impact of group heterogeneity. Probabilities of fluctuations as functions of* δ *for the three methods of Section 2.*

for Method I. For joint events in (2.5) and (2.6) for Methods II and III, $\alpha_2 = \alpha_3 = 0.1$, to have a fair comparison of the three methods. However, the last two examples have $\alpha_2 = \alpha_3 = 0.05$, to illustrate a point stated a few paragraphs below. Precision thresholds are $c = k = 0.05$.

Table 1 gives the maximum credibility factors Z that satisfy the corresponding limited-fluctuation credibility conditions. The first six scenarios reveal an agreement between the real data and the prior information, in the sense of Section 2. Disagreement is introduced in the next three scenarios. The last two cases illustrate that all the three methods converge to the classical case when the prior distribution is assumed known with no error.

Full credibility $Z = 1$ is assigned in scenario 1, when there is a sufficiently high frequency of claims λ under a low severity standard deviation σ. Scenarios 2-6 are modifications of scenario 1. A high prior variance in scenario 2 has no impact on the full credibility because the compromise estimator is solely based on the real data in this case. On the other hand, a lower frequency λ in scenario 3 and a higher loss standard deviation σ in scenario 4 increase the degree of uncertainty in the real data. As a result, only a partial credibility can be assigned. In the case of both a lower frequency and a higher loss standard deviation, as in scenario 5, we reach the critical level of uncertainty where no credibility is possible. This situation can be rescued when the uncertainty in real data is compensated by a more reliable

Table 1: Credibility Factor under Several Scenarios

Scenario	Real Data Parameters	Hyper Parameters	Degree of Disagreement	Method I	Method II	Method III
1	$\theta = 200$, $\sigma = 40$ $\lambda = 600$	$\nu = 120,000$ $\tau = 10,000$	$\delta = 0$	$Z = 1$	$Z = 1$	$Z = 1$
2	$\theta = 200$, $\sigma = 40$ $\lambda = 600$	$\nu = 120,000$ $\tau = 50,000$	$\delta = 0$	$Z = 1$	$Z = 1$	$Z = 1$
3	$\theta = 200$, $\sigma = 40$ $\lambda = 360$	$\nu = 72,000$ $\tau = 10,000$	$\delta = 0$	$Z = 0.822$	$Z = 0.980$	$Z = 0.971$
4	$\theta = 200$, $\sigma = 180$ $\lambda = 600$	$\nu = 120,000$ $\tau = 10,000$	$\delta = 0$	$Z = 0.804$	$Z = 0.959$	$Z = 0.99$
5	$\theta = 200$, $\sigma = 180$ $\lambda = 360$	$\nu = 72,000$ $\tau = 10,000$	$\delta = 0$	no credibility	no credibility	no credibility
6	$\theta = 200$, $\sigma = 180$ $\lambda = 360$	$\nu = 72,000$ $\tau = 3,000$	$\delta = 0$	$Z = 0.623$	$Z = 0.743$	$Z = 0.653$
1a	$\theta = 200$, $\sigma = 40$ $\lambda = 600$	$\nu = 124,000$ $\tau = 10,000$	$\delta = 0.4$	$Z = 1$	$Z = 1$	$Z = 1$
3a	$\theta = 200$, $\sigma = 40$ $\lambda = 360$	$\nu = 76,000$ $\tau = 10,000$	$\delta = 0.4$	no credibility	$Z = 0.980$	$Z = 0.965$
6a	$\theta = 200$, $\sigma = 180$ $\lambda = 360$	$\nu = 73,200$ $\tau = 3,000$	$\delta = 0.4$	$Z = 0.623$	$Z = 0.743$	$Z = 0.596$
3b	$\theta = 200$, $\sigma = 40$ $\lambda = 360$	$\nu = 72,004$ $\tau = 10$	$\delta = 0.4$	$Z = 0.822$	$Z = 0.822$	$Z = 0.822$
6b	$\theta = 200$, $\sigma = 180$ $\lambda = 360$	$\nu = 72,004$ $\tau = 10$	$\delta = 0.4$	$Z = 0.623$	$Z = 0.623$	$Z = 0.623$

prior information with a low prior standard deviation τ, as in scenario 6.

Disagreement, in the sense of Section 3, does not affect the full credibility in scenario 1a, where estimation of the pure premium is based on the real data without using the prior distribution. On the other hand, disagreement may reduce the partial credibility. For Method I, the maximum partial credibility factor Z is determined by probability $p_R \leq \alpha_R$ in (2.3). If the prior satisfies its limited-fluctuation condition $p_H \leq \alpha_H$, then partial credibility can be assigned, as in scenario 6a. Otherwise, there is no credibility factor Z that satisfies both conditions in (2.3), and this case results in no credibility, as in scenario 3a. Scenarios 1a, 3a, and 6a are blueprints of scenarios 1, 3, and 6, only with some disagreement between the real data and the prior.

The last two examples 3b and 6b repeat scenarios 3a and 6a with the only difference – practically no uncertainty in the prior distribution, notice a very low prior standard deviation $\tau = 10$. We observe two facts here. The "no credibility" situation in scenario 3a is now corrected in scenario 3b. In general, credibility estimation is *always possible* when the prior distribution is known without any error (perhaps, unrealistically), because even in the extreme case of no history of real data, the option of $Z = 0$ is still available, and the pure premium is estimated from the prior only. Second, we see that in the case of a completely known prior distribution, all three methods converge to the same credibility, and it coincides with the classical partial credibility factor in Klugman et al. (2012) or Herzog (2015).

Densities of actual data and the prior mean used in Scenarios 1 and 1a are shown on Figure 1.

5. Summary and conclusions

According to Actuarial Standards of Practice No. 25 (2011), "The actuary should apply credibility procedures that appropriately consider the characteristics of both the subject experience and the relevant experience". In actuarial practice, characteristics of real data (subject experience) are typically fully addressed by developing and fitting appropriate loss models. At the same time, uncertainty of the prior distribution (relevant experience) are often neglected, and the prior is assumed to be absolutely known without an error.

In this article, we propose three methods, three limited-fluctuation conditions that account for the uncertainty of the prior. It is concluded that:

- Uncertainty of the real data as well as uncertainty of the prior affect credibility factors and decisions concerning full and partial credibility. Credibility increases with the expected frequency of losses.

- When much uncertainty exists in both the real data and the prior, there may be a case when no credibility can be assigned, even a partial one. A larger experience or a more informative prior are required to assign credibility. This situation will not exist when the prior is considered fully credible, with no uncertainty.

- Accounting for heterogeneity of risk groups, credibility also depends on the location of the insured's expected loss relative to the prior expectation. The current experience is more credible if the actual expected loss agrees with the prior.

Examples of the previous section suggest that all three methods essentially agree on their assignment of high/full or low/no credibility. Indeed, in order to obtain a credible estimator of the expected total loss, both the real and the hypothetical components (R and H, $\overline{X}$ and μ, sample and prior, individual and collective experience, etc.) must be credible, and the three methods express these requirements in three different ways.

Method I expresses this credibility separately for R and H. For example, if one of the two components fails to be credible, an estimator based solely on the other component may be considered, giving it full credibility. This can happen when the real history of an insured is too short to contain any events, or when the prior expectation is "purely a guess", as it is put by Tindall and Mast (2003). In such cases, one should see if there is an acceptable combination of c, k, α_R, and α_H that leads to an interval in (2.4) that contains $Z = 0$, for the full credibility of the prior, or $Z = 1$, for the full credibility of the real data. Then the credibility problem still has a solution.

Methods II and III impose *joint* conditions on the credibility of real and hypothetical data. In general, these conditions may be considered milder because a high credibility of one source of data may compensate for a low credibility of the other, a feature that Method I does not have. Method III does not consider credibility of R and H at all. Instead, it goes directly to the compromise estimator and states the low-fluctuation credibility criterion in terms of its precision. If this is the only goal, Method III is appropriate. However, if credibility can be improved by expanding one or the other source of data or enhancing the quality of such data, one may find Methods I and II appropriate for this approach.

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