Ruin Probability-Based Initial Capital of the Discrete-Time Surplus Process

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ABSTRACT

This paper studies an insurance model under the regulation that the insurance company has to reserve sufficient initial capital to ensure that ruin probability does not exceed the given quantity α. We prove the existence of the minimum initial capital. To illustrate our results, we give an example in approximating the minimum initial capital for exponential claims.

KEYWORDS

Initial capital, insurance, claim process, ruin probability

1. Introduction

In recent years, risk models have attracted much attention in the insurance business, in connection with the possible insolvency and the capital reserves of insurance companies. The main interest from the point of view of an insurance company is claim arrival and claim size, which affect the capital of the company.

In this paper, we assume that all the processes are defined in a probability space (Ω, F, Pr). Claims happen at the times T_i, satisfying $0 = T_0 \le T_1 \le T_2 \le \ldots$. The nth claim arriving at time T_n causes the claim size X_n. Now let the constant c represent the premium rate for one unit time; the random variable cT_n describes the inflow of capital into the business by time T_n, and $\sum_{i=1}^{n} X_i$ describes the outflow of capital due to payments for claims occurring in $[0, T_n]$. Therefore, the quantity

$$U_0 = u, U_n = u + cT_n - \sum_{i=1}^{n} X_i \tag{1.1}$$

is the insurer's balance (or surplus) at time T_n, $n = 1, 2, 3, \ldots$, with the constant $U_0 = u \ge 0$ as the initial capital.

We consider the discrete-time surplus process (1.1) in the situation that the possible insolvency (ruin) can occur only at claim arrival times $T_n = n$, $n = 1, 2, 3, \ldots$. Thus, the model 1.1 becomes

$$U_0 = u, U_n = u + cn - \sum_{i=1}^{n} X_i \tag{1.2}$$

for all $n = 1, 2, 3, \ldots$.

The general approach for studying ruin probability in the discrete-time surplus process is through the so-called *Gerber-Shiu discounted penalty function;* for example, Pavlova and Willmot (2004), Dickson (2005), and Li (2005a, 2005b). All of these articles study (or calculate) the ruin probability as a function the initial capital $u \ge 0$. In this paper, we shall work in the opposite direction, i.e., we study the initial capital for the discrete-time surplus process as a function of ruin probabilities.

2. Main results

Let $\{U_n, n \ge 0\}$ be a surplus process (as in Section 1) that is driven by the claim process $\{X_n, n \ge 1\}$. We consider the finite-time ruin probabilities of the discrete-time surplus process in Equation (1.2) with the independent and identically distributed (i.i.d.) claim process $\{X_n, n \ge 1\}$. We let $F_{X_1}(x)$ be the distribution function of X_1, i.e.,

$$F_{X_1}(x) = \Pr\{X_1 \le x\}. \tag{2.1}$$

The *premium rate c* is calculated by the *expected value principle,* i.e.,

$$c = (1 + \theta) E[X_1] \tag{2.2}$$

where $\theta > 0$ which is the *safety loading of insurer.*

Let $u \ge 0$ be an initial capital. For each $n = 1, 2, 3, \ldots$, we let

$$\varphi_n(u) := \Pr\{U_1 \ge 0, U_2 \ge 0, U_3 \ge 0, \ldots, U_n \ge 0 | U_0 = u\} \tag{2.3}$$

denote the *survival probability* at the times n. Thus, the *ruin probability* at one of the time 1, 2, 3, . . . , n is denoted by

$$\Phi_n(u) = 1 - \varphi_n(u). \tag{2.4}$$

Definition 2.1. Let $\{U_n, n \ge 0\}$ be a surplus process which is driven by the claim process $\{X_n, n \ge 1\}$ and $c > 0$ be a premium rate. Given $\alpha \in (0, 1)$ and $N \in \{1, 2, 3, \ldots\}$. Let an initial capital $u \ge 0$, if $\Phi_N(u) \le \alpha$ then u is called an *acceptable initial capital* corresponding to (α, N, c, $\{X_n, n \ge 1\}$). Particularly, if

$$u^* = \min_{u \ge 0}\{u : \Phi_N(u) \le \alpha\} \tag{2.5}$$

exists, u^* is called the *minimum initial capital* corresponding to (α, N, c, $\{X_n, n \ge 1\}$) and is written as

$$u^* := \text{MIC}(\alpha, N, c, \{X_n, n \ge 1\}). \tag{2.6}$$

2.1. Ruin and survival probability

We define the *total claim process* by

$$S_n := X_1 + X_2 + \cdots + X_n \tag{2.7}$$

for all $n = 1, 2, 3, \ldots$.

Lemma 2.1. Let $N \in \{1, 2, 3, \ldots\}$ and $c > 0$ be given. If $\{X_n, n \geq 1\}$ is an i.i.d. claim process, then $\varphi_N(u)$ is increasing and right continuous and $\Phi_N(u)$ is decreasing and right continuous in u.

Proof. The survival probability at the time N as mentioned in (2.3) can be expressed as follows.

$$\begin{aligned}\varphi_N(u) &= \Pr\{S_1 \leq u + c, S_2 \leq u \\ &\quad + 2c, \ldots, S_N \leq u + Nc\} \\ &= \Pr\left(\bigcap_{k=1}^{N}\{S_k - kc - u \leq 0\}\right) \\ &= E\left[\mathbb{I}_{\bigcap_{k=1}^{N}\{S_k - kc - u \leq 0\}}\right] \\ &= E\left[\prod_{k=1}^{N}\mathbb{I}_{\{S_k - kc - u \leq 0\}}\right] \end{aligned} \tag{2.8}$$

where

$$\mathbb{I}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

for all $A \subset \mathbb{R}$. Since $\mathbb{I}_{\{S_k - kc - u \leq 0\}}(\omega) = \mathbb{I}_{(-\infty,0]}(S_k(\omega) - kc - u)$ for all $\omega \in \Omega$,

$$\varphi_N(u) = E\left[\prod_{k=1}^{N}\mathbb{I}_{(-\infty,0]}(S_k - kc - u)\right]. \tag{2.9}$$

For each $a \in \mathbb{R}$ and $u \geq 0$, we obtain

$$\mathbb{I}_{(-\infty,0]}(a - u) = \begin{cases} 1, & u \geq a \\ 0, & u < a, \end{cases}$$

then $\mathbb{I}_{(-\infty,0]}(a - u)$ is increasing and right continuous in u. This implies that $\Pi_{i=1}^{N}\mathbb{I}_{(-\infty,0]}(a_i - u)$ is also increasing and right continuous in u, moreover, this bounding function is identically equal to 1, where $a_k \in \mathbb{R}$, $k = 1, 2, 3, \ldots, N$. Therefore, by the monotone convergence theorem, we have

$$\begin{aligned}\lim_{v \to u^+}\varphi_N(v) &= \lim_{v \to u^+} E\left[\prod_{k=1}^{N}\mathbb{I}_{(-\infty,0]}(S_k - kc - v)\right] \\ &= E\left[\lim_{v \to u^+}\prod_{k=1}^{N}\mathbb{I}_{(-\infty,0]}(S_k - kc - v)\right] \\ &= E\left[\prod_{k=1}^{N}\mathbb{I}_{(-\infty,0]}(S_k - kc - u)\right] \\ &= \varphi_N(u). \end{aligned} \tag{2.10}$$

Therefore, $\varphi_N(u)$ is increasing and right continuous. Moreover, we can conclude that $\Phi_N(u) = 1 - \varphi_N(u)$ is decreasing and also right continuous.

Theorem 2.2. Let $N \in \{1, 2, 3, \ldots\}$ and $c > 0$ be given. If $\{X_n, n \geq 1\}$ is an i.i.d. claim process, then

$$\lim_{u \to \infty}\varphi_N(u) = 1 \text{ and } \lim_{u \to \infty}\Phi_N(u) = 0. \tag{2.11}$$

Proof. First, we will show the following properties

$$\bigcap_{i=1}^{N}\{X_i \leq u + c\} \subset \bigcap_{i=1}^{N}\{S_i \leq Nu + ic\}. \tag{2.12}$$

Let $\omega \in \bigcap_{i=1}^{N}\{X_i \leq u + c\}$ be given. For each $i \in \{1, 2, 3, \ldots, N\}$, we have $X_i(\omega) \leq u + c$ and

$$S_i(\omega) = \sum_{k=1}^{i} X_k(\omega) \leq iu + ic \leq Nu + ic. \tag{2.13}$$

That is, $\omega \in \{S_i \leq Nu + ic\}$. Therefore, (2.12) follows. Next, since the process $\{X_n, n \geq 1\}$ is i.i.d., then

$$\Pr\left(\bigcap_{i=1}^{N}\{X_i \leq u + c\}\right) = \prod_{i=1}^{N}\Pr\{X_i \leq u + c\} = (F(u + c))^N. \tag{2.14}$$

By Equation (2.8), we have

$$\varphi_N(Nu) = \Pr\left(\bigcap_{i=1}^{N}\{S_i \leq Nu + ic\}\right). \tag{2.15}$$

By (2.12), (2.14) and (2.15), we obtain

$$(F(u + c))^N \leq \varphi_N(Nu) \leq 1. \tag{2.16}$$

Since $(F(u+c))^N \to 1$ as $u \to \infty$, then $\varphi_N(Nu) \to 1$ as $u \to \infty$. Thus, we conclude that $\varphi_N(u) \to 1$, and $\Phi_N(u) = 1 - \varphi_N(u) \to 0$ as $u \to \infty$. This is the proof.

Corollary 2.3. Let $\alpha \in (0, 1)$, $N \in \{1, 2, 3, \ldots\}$ and $c > 0$ be given. If $\{X_n, n \geq 1\}$ is an i.i.d. claim process, then there exists $\tilde{u} \geq 0$ such that, for all $u \geq \tilde{u}$, u is an acceptable initial capital corresponding to $(\alpha, N, c, \{X_n, n \geq 1\})$.

Proof. We consider by case. Case 1: $\Phi_N(0) \leq \alpha$. Since $\Phi_N(u)$ is decreasing, then $\Phi_N(u) \leq \Phi_N(0) \leq 0$ for all $u \geq 0$. Case 2: $\Phi_N(0) > \alpha$. By Theorem 2.2, we have $\Phi_N(u) \to 0$ as $u \to \infty$. Thus, there exists $\tilde{u} > 0$ such that $\Phi_N(\tilde{u}) < \alpha$. Since $\Phi_N(u)$ is decreasing, we conclude that $\Phi_N(u) \leq \Phi_N(\tilde{u}) < \alpha$ for all $u \geq \tilde{u}$.

2.2. Recursive formula of ruin probabilities

From Theorem 2.2 and Corollary 2.3, we know that the small ruin probability can be controlled by a large initial capital. In this part, we shall describe the upper bound of ruin probability with negative exponential. In order to prove the following lemma, we shall use an equivalent definition of the ruin probability which is given as follows:

$$\Phi_n(u) = \Pr\left(\max_{1\leq k\leq n}\left(\sum_{i=1}^{k} X_i - ck\right) > u\right). \tag{2.17}$$

Theorem 2.4. Let $N \in \{1, 2, 3, \ldots\}$, $c > 0$ and $u \geq 0$ be given. If $\{X_n, n \geq 1\}$ is an i.i.d. claim process, then the ruin probability at one of the times $1, 2, 3, \ldots, N$ satisfies the following equation

$$\Phi_N(u) = \Phi_1(u) + \int_{-\infty}^{u+c} \Phi_{N-1}(u+c-x)\,dF_{X_1}(x) \tag{2.18}$$

where $\Phi_0(u) = 0$.

Proof. We will prove (2.18) by induction. We start with $n = 1$. Since $\Phi_0(u) = 0$ for all $u \geq 0$, then

$$\int_{-\infty}^{u+c} \Phi_0(u+c-x)\,dF_{X_1}(x) = 0. \tag{2.19}$$

This proves (2.18) for $n = 1$. Now assume that (2.18) holds for $n = k \geq 1$. Then

$$\begin{aligned}
\Phi_{k+1}(u) &= \Pr\left(\max_{1\leq n\leq k+1}\left(\sum_{i=1}^{n} X_i - cn\right) > u\right)\\
&= \Pr(X_1 - c > u)\\
&\quad + \Pr\left(\max_{2\leq n\leq k+1}\left(X_1 + \sum_{i=2}^{n} X_i - cn\right) > u, X_1 \leq u+c\right)\\
&= \Phi_1(u) + \int_{-\infty}^{u+c} \Pr\left(\max_{1\leq n\leq k}\left(x + \sum_{i=2}^{n} X_i - cn\right) > u\right)\\
&\quad dF_{X_1}(x)\\
&= \Phi_1(u) + \int_{-\infty}^{u+c} \Pr\left(\max_{2\leq n\leq k+1}\left(\sum_{i=2}^{n} X_i - c(n-1)\right)\right.\\
&\quad \left. > u+c-x\right) dF_{X_1}(x)\\
&= \Phi_1(u) + \int_{-\infty}^{u+c} \Pr\left(\max_{2\leq n\leq k+1}\left(\sum_{i=1}^{n-1} X_i - c(n-1)\right)\right.\\
&\quad \left. > u+c-x\right) dF_{X_1}(x)\\
&= \Phi_1(u) + \int_{-\infty}^{u+c} \Pr\left(\max_{1\leq n\leq k}\left(\sum_{i=1}^{n} X_i - cn\right)\right.\\
&\quad \left. > u+c-x\right) dF_{X_1}(x)\\
&= \Phi_1(u) + \int_{-\infty}^{u+c} \Phi_k(u+c-x)\,dF_{X_1}(x).
\end{aligned}$$

which proves (2.18) for $n = k + 1$ and concludes the proof.

Corollary 2.5. Let $N \in \{1, 2, 3, \ldots\}$, $c > 0$ and $u \geq 0$ be given. If $\{X_n, n \geq 1\}$ is an i.i.d. claim process, then the ruin probability at one of the times $1, 2, 3, \ldots, N$ satisfies the following equation:

$$\begin{aligned}
&\Phi_0(u) = 0, \Phi_1(u) = 1 - \Pr(X \leq u+c), \Phi_N(u)\\
&\quad = \Phi_{N-1}(u) + \Theta_N(u)
\end{aligned} \tag{2.20}$$

where

$$\Theta_N(u) = \int_{-\infty}^{u+c}\left(\int_{-\infty}^{u+c-x} \Phi_{N-2}(u+2c-x-v)\,dF_{X_1}(v)\right) dF_{X_1}(x)$$

for all $n = 2, 3, 4, \ldots$.

Proof. Let $N \geq 2$, by Theorem 2.4, we obtain

$$\Phi_N(u) = \Phi_1(u) + \int_{-\infty}^{u+c} \Phi_{N-1}(u+c-x)\,dF_{X_1}(x)$$
$$= \Phi_1(u) + \int_{-\infty}^{u+c} \Big(\Phi_{N-2}(u+c-x) + \int_{-\infty}^{u+2c-x} \Phi_{N-2}(u+2c-x-v)\,dF_{X_1}(v)\Big)dF_{X_1}(x)$$
$$= \Phi_1(u) + \int_{-\infty}^{u+c} (\Phi_{N-2}(u+c-x)\,dF_{X_1}(x) + \int_{-\infty}^{u+c}\Big(\int_{-\infty}^{u+2c-x} \Phi_{N-2}(u+2c-x-v)\,dF_{X_1}(v)\Big)dF_{X_1}(x)$$
$$= \Phi_{N-1}(u) + \int_{-\infty}^{u+c}\Big(\int_{-\infty}^{u+2c-x} \Phi_{N-2}(u+2c-x-v)\,dF_{X_1}(v)\Big) dF_{X_1}(x).$$

This completes the proof.

Corollary 2.6. Let $N \in \{1, 2, 3, \ldots\}$ and $u \geq 0$. Assume that $\{X_n, n \geq 1\}$ is a sequence of exponential distribution with intensity $\lambda > 0$, i.e., X_1 has the probability density function $f(x) = \lambda e^{-\lambda x}$. The obtained ruin probability is in the following recursive form

$$\Phi_0(u) = 0, \Phi_n(u) = \Phi_{n-1}(u) + \frac{(u+c)\lambda^{n-1}(u+nc)^{n-2}}{(n-1)!}e^{-\lambda(u+nc)} \tag{2.21}$$

for all $n = 1, 2, 3, \ldots$, where the initial capital $u \geq 0$ and premium rate $c > E[X_1] = 1/\lambda$.

Proof. We will prove (2.21) by induction. We start with $n = 1$, $\Phi_1(u) = 1 - \Pr\{X \leq u + c\} = 1 - (1 - e^{-\lambda(u+c)}) = e^{-\lambda(u+c)}$.

This proves (2.21) for $n = 1$. Next we assume that (2.21) holds for $n = k \geq 1$. From Theorem 2.2, we have

$$\Phi_{k+1}(u) = \Phi_1(u) + \int_0^{u+c} \Phi_k(u+c-x)\,dF_{X_1}(x)$$
$$= \Phi_1(u) + \int_0^{u+c}\Big(\Phi_{k-1}(u+c-x) + \frac{(u+2c-x)\lambda^{k-1}(u+(k+1)c-x)^{k-2}}{(k-1)!} e^{-\lambda(u+(k+1)c-x)}\Big)dF_{X_1}(x)$$
$$= \Phi_k(u) + \int_0^{u+c} \frac{(u+2c-x)\lambda^{k-1}(u+(k+1)c-x)^{k-2}}{(k-1)!} e^{-\lambda(u+(k+1)c-x)}dF_{X_1}(x) \tag{2.22}$$

and

$$\Theta_{k+1}(u) = \int_0^{u+c} \frac{(u+2c-x)\lambda^{k-1}(u+(k+1)c-x)^{k-2}}{(k-1)!} e^{-\lambda(u+(k+1)c-x)}\lambda e^{-\lambda x}dx$$
$$= \frac{\lambda^k e^{-\lambda(u+(k+1)c)}}{(k-1)!}\int_0^{u+c}(u+(k+1)c-x)^{k-2}(u+(k+1)c-x-(k-1)c)\,dx$$
$$= \frac{\lambda^k e^{-\lambda(u+(k+1)c)}}{(k-1)!}\int_0^{u+c}((u+(k+1)c-x)^{k-1} + (k-1)c(u+(k+1)c-x)^{k-2})\,dx$$
$$= \frac{(u+c)\lambda^k(u+(k+1)c)^{k-1}}{k!}e^{-\lambda(u+(k+1)c)},$$

which proves (2.21) for $n = k + 1$ and completes the proof.

2.3. Existence of minimum initial capital

A quantity α, which was discussed in the previous section, can be described as the most acceptable probability that the insurance company will become insolvent. As a result of Corollary 2.3, we obtain that $\{u \geq 0 : \Phi_N(u) \leq \alpha\}$ is a non-empty set. This means that we can always choose an initial capital which makes the value of ruin probability not exceed α. Since the set $\{u \geq 0 : \Phi_N(u) \leq \alpha\}$ is an infinite set, then there are many acceptable initial capital corresponding to $(\alpha, N, c, \{X_n, n \geq 1\})$. In this section, we will prove the existence of

$$\mathrm{MIC}(\alpha, N, c, \{X_n, n \geq 1\}) = \min_{u \geq 0}\{u : \Phi_N(u) \leq \alpha\}. \tag{2.23}$$

Lemma 2.2. Let a, b and α be real numbers such that $a \leq b$. If f is decreasing and right continuous on $[a, b]$ and $\alpha \in [f(b), f(a)]$, then there exists $d \in [a, b]$ such that

$$d = \min\{x \in [a, b] : f(x) \leq \alpha\}. \tag{2.24}$$

Proof. Let

$$S := \{x \in [a,b] : f(x) \le \alpha\}.$$

Since $\alpha \in [f(b), f(a)]$, i.e., $f(b) \in \alpha \le f(a)$, then $b \in S$. That is, S is a non-empty set. Since S is a subset of closed and bounded interval $[a, b]$, then there exists $d \in [a, b]$ such that $d = \inf S$. Next, we consider by case.

Case 1: $d = b$. We know that $b \in S$, thus $b = \min S$.

Case 2: $a \le d < b$. Since $d = \inf S$, then there exists $d_n \in S$ such that

$$d \le d_n < d + 1/n$$

for all $n \in \mathbb{N}$. For each $n > 2/(b - d)$, we have

$$d < d + 1/n < d + \frac{b-d}{2} = \frac{b+d}{2} < b.$$

This means that $d + 1/n \in (d, b) \subset [a, b]$ for all $n > 2/(b - d)$. Since f is decreasing and $d_n \in S$, we get

$$f(d+1/n) \le f(d_n) \le \alpha,$$

i.e., $d + 1/n \in S$ for all $n > 2/(b - d)$. Since f is right continuous at d, we have

$$f(d) = \lim_{n\to\infty} f(d+1/n) \le \alpha.$$

Therefore, $d \in S$, i.e., $d = \min S$. This completes the proof.

Theorem 2.7. Let $\alpha \in (0, 1)$, $N \in \{1, 2, 3, \ldots\}$, and $c > 0$. Then there exist $u^* \ge 0$ such that

$$u^* = \mathrm{MIC}(\alpha, N, c, \{X_n, n \ge 1\}).$$

Proof. We consider by case. Case 1: $\Phi_N(0) \le \alpha$, we have

$$\mathrm{MIC}(\alpha, N, c, \{X_n, n \ge 1\}) = 0.$$

Case 2: $\Phi_N(0) > \alpha$, by Corollary 2.3, there exists $\tilde{u} > 0$ such that $\Phi_N(\tilde{u}) < \alpha$, i.e., $\alpha \in [\Phi_N(\tilde{u}), \Phi_N(0)]$. Since $\Phi_N(u)$ is decreasing and right continuous, by Lemma 2.2 there exists $u^* \in [0, \tilde{u}]$ such that

$$u^* = \min_{u\in[0,\tilde{u}]} \{u : \Phi_N(u) \le \alpha\} = \min_{u\in[0,\infty)} \{u : \Phi_N(u) \le \alpha\}.$$

That is,

$$u^* = \mathrm{MIC}(\alpha, N, c, \{X_n, n \ge 1\}).$$

Next, we will approximate the minimum initial capital $\mathrm{MIC}(\alpha, N, c, \{X_n, n \ge 1\})$ by applying the bisection technique for the decreasing and right continuous function.

Theorem 2.8. Let $\alpha \in (0, 1)$, $N \in \{1, 2, 3, \ldots\}$, and $v_0, u_0 \ge 0$ such that $v_0 < u_0$. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be a real sequence defined by

$$\begin{cases} v_k = v_{k-1} \quad \text{and} \quad u_k = \dfrac{u_{k-1} + v_{k-1}}{2}, & \text{if } \Phi_N\left(\dfrac{u_{k-1} + v_{k-1}}{2}\right) \le \alpha \\ v_k = \dfrac{v_{k-1} + u_{k-1}}{2} \quad \text{and} \quad u_k = u_{k-1}, & \text{if } \Phi_N\left(\dfrac{u_{k-1} + v_{k-1}}{2}\right) > \alpha \end{cases}$$

for all $k = 1, 2, 3, \ldots$. If $\Phi_N(u_0) \le \alpha < \Phi_N(v_0)$, then

$$\lim_{k\to\infty} u_k = \mathrm{MIC}(\alpha, N, c, \{X_n, n \ge 1\}) \tag{2.25}$$

and

$$0 \le u_k - \mathrm{MIC}(\alpha, N, c, \{X_n, n \ge 1\}) \le \frac{u_0 - v_0}{2^k} \tag{2.26}$$

for all $k = 1, 2, 3, \ldots$.

Proof. Obviously, $\{u_n\}_{n=1}^{\infty}$ is decreasing and $\{v_n\}_{n=1}^{\infty}$ is increasing, moreover, $v_k \le u_k$ for all $k = 1, 2,$

Table 1. Minimum initial capital MIC (α, *N*, *c*, $\{X_n, n \geq 1\}$) in the discrete-time surplus process with exponential claims ($\lambda = 1$)

	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$	
N	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.10$	$\theta = 0.25$
10	4.31979	3.39733	2.89299	2.09364	1.99866	1.29821
20	5.80757	4.13270	3.98629	2.58739	2.84099	1.65474
30	6.79110	4.47565	4.69130	2.80479	3.37378	1.80597
40	7.52286	4.66050	5.20540	2.91736	3.75643	1.88242
50	8.09889	4.76749	5.60309	2.98061	4.04866	1.92467
100	9.81693	4.92644	6.74520	3.07093	4.86621	1.98377
200	11.13546	4.94953	7.56253	3.08341	5.42576	1.99174
300	11.60284	4.95021	7.83409	3.08377	5.60493	1.99197
400	11.79769	4.95024	7.94308	3.08378	5.67545	1.99197
500	11.88611	4.95024	7.99136	3.08378	5.70634	1.99197
1,000	11.96919	4.95024	8.03565	3.08378	5.73435	1.99197
5,000	11.97291	4.95024	8.03757	3.08378	5.73554	1.99197
10,000	11.97291	4.95024	8.03757	3.08378	5.73554	1.99197

3, ... Thus, $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are convergent. Since

$$0 \leq u_k - v_k = (u_0 - v_0)/2^k \to 0 \text{ as } k \to \infty,$$

then there exists $u^* \in [v_0, u_0]$ such that

$$\lim_{k\to\infty} u_k = \lim_{k\to\infty} v_k := u^*. \tag{2.27}$$

Since $\Phi_N(u)$ is decreasing and $\Phi_N(v_k) > \alpha$ for all $k = 1, 2, 3, \ldots$, then $\Phi_N(u) > \alpha$ for all $u < u^*$. Since $\Phi_N(u)$ is right continuous and $\Phi_N(u_k) \leq \alpha$ for all $k = 1, 2, 3, \ldots$, then

$$\Phi_N(u^*) = \lim_{k\to\infty} \Phi_N(u_k) \leq \alpha. \tag{2.28}$$

Therefore,

$$u^* = \text{MIC}(\alpha, N, c, \{X_n, n \geq 1\}). \tag{2.29}$$

For each $k = 0, 1, 2, \ldots$, we have $v_k \leq u^* \leq u_k$. This implies that

$$\begin{aligned} 0 \leq u_k - u^* &\leq u_k - u^* + u^* - v_k \\ &= u_k - v_k = \frac{u_0 - v_0}{2^k}. \end{aligned} \tag{2.30}$$

This completes the proof.

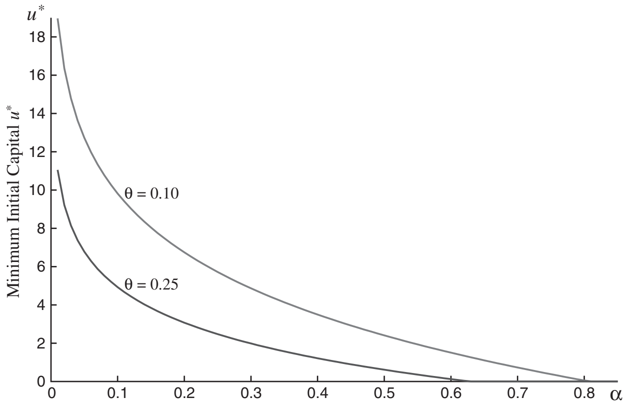
2.4. Numerical results

We provide numerical illustrations of the main results. We approximate the minimum initial capital of the discrete-time surplus process (1.2) by using Theorem 2.8 in the case of $\{X_n, n \geq 1\}$ a sequence of i.i.d. exponential distribution with intensity $\lambda = 1$, by choosing model parameter combinations $\theta = 0.10$ and 0.25, i.e., $c = 1.10$ and $c = 1.25$, respectively; and $\alpha = 0.1$, 0.2, and 0.3.

Table 1 shows the approximation of MIC(α, *N*, *c*, $\{X_n, n \geq 1\}$) with u_{25} as mentioned in Theorem 2.8, choosing $v_0 = 0$ and $u_0 = 20$, and $\Phi_N(u)$ is computed from the recursive form (2.21).

Figure 1 shows the approximation of MIC(α, *N*, *c*, $\{X_n, n \geq 1\}$) for the various values of α with u_{25} as mentioned in Theorem 2.8. Here we choose $v_0 = 0$, $u_0 = 20$, and parameter combinations $\theta = 0.10$, $\theta = 0.25$, i.e., $c = 1.10$ and $c = 1.25$, respectively.

Figure 1. Minimum initial capital MIC(α, N, c, $\{X_n, n \geq 1\}$) in the discrete-time surplus process with exponential claims ($\lambda = 1$, $N = 100$)



References

Dickson, D. C. M., *Insurance Risk and Ruin.* New York: Cambridge University Press, 2005.

Li, S., "On a Class of Discrete Time Renewal Risk Models," *Scandinavian Actuarial Journal* 2005a(4), pp. 241–260.

Li, S., "Distributions of the Surplus before Ruin, the Deficit at Ruin and the Claim Causing Ruin in a Class of Discrete Time Risk Model," *Scandinavian Actuarial Journal* 2005b (2), pp. 271–284.

Pavlova, K. P., and G. E. Willmot, "The Discrete Stationary Renewal Risk Model and the Gerber-Shiu Discounted Penalty Function," *Insurance: Mathematics and Economics* 35(2): 2004, pp. 267–277.