One-Year and Total Run-Off Reserve Risk Estimators Based on Historical Ultimate Estimates

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ABSTRACT

This paper presents closed-form formulas in order to estimate, based on the historical triangle of ultimate estimates, both the one-year and the total run-off reserve risk. This is helpful in case (as is often usual in practice) the reserve risk formulas related to the applied reserving methodology are unknown or in case such formulas cannot be rigorously derived since a fully well-defined stochastic model supporting the reserving methodology is missing (e.g., due to mixing of reserving methods).

KEYWORDS

Reserve risk, mean square error of prediction, one-year prediction uncertainty, total run-off prediction uncertainty, ultimate estimates

1. Introduction

Within the stochastic claims reserving theory, it is good practice to derive estimators for both the one-year and the total run-off reserve risk (prediction uncertainty) whenever a new reserving model is defined. This is usually done considering the appropriate conditional mean square error of prediction (MSEP).

For example, for the standard chain-ladder (CL) methodology, which is the most popular reserving methodology and is generally supported by Mack's (1993) distribution-free model, the total run-off prediction uncertainty estimator is usually given by Mack's formula, and the one-year prediction uncertainty estimator is given by Merz and Wüthrich's (2008) formula.

Unfortunately, such estimators have not yet been derived in a closed form for all existing stochastic reserving models, and for many reserving methodologies used in practice within the insurance industry, a fully well-defined supporting stochastic model is missing (often because of a mixture of basic methods). Therefore, the reserve risk estimators cannot be rigorously derived.

The lack of such formulas is currently an open problem in the insurance industry (see Dal Moro and Lo 2014). The central question is how to quantify the prediction uncertainties (both one-year and total run-off) if one is faced with the above-stated situation.

In this paper I suggest an answer to this question. As already proposed in Rehman and Klugman (2010), I will take the following unusual approach that also allows to implicitly account for model risk associated with the choice of an appropriate reserving methodology:

I do not specify the stochastic reserving model supporting the reserving methodology (since, as already mentioned, this cannot always be well defined) but only assume a stochastic model for the ultimate estimates. That allows one to derive an estimator for the conditional MSEP of the prediction uncertainties (both one-year and total run-off) based on the historical triangle of ultimate estimates only, i.e., regardless of which underlying reserving methodology generates the ultimate estimates.

Table 1. Ultimate estimates ($\hat{U}_{i,j}$)

i/j	0	1	$\cdots$	$I-1$	I
0	$\hat{U}_{0,0}$	$\hat{U}_{0,1}$	$\cdots$	$\hat{U}_{0,I-1}$	$\hat{U}_{0,I}$
1	$\hat{U}_{1,0}$	$\hat{U}_{1,1}$	$\cdots$	$\hat{U}_{1,I-1}$	
$\vdots$	$\vdots$	$\vdots$			
$I-1$	$\hat{U}_{I-1,0}$	$\hat{U}_{I-1,1}$			
I	$\hat{U}_{I,0}$				

In Table 1, I denote with $\hat{U}_{i,j}$ the estimated ultimate claims amount for accident year $i \in \{0, \ldots, I\}$ at development period $j \in \{0, \ldots, I\}$.

Basically, the stochastic model for the ultimate estimates presented in this paper assumes that for any accident year i, the estimated ultimate losses of two consecutive development periods j and $j+1$ are related by a proportionality factor g_j, which depends on only the development period j, i.e., it holds true that

$$\hat{U}_{i,j+1} \approx g_j \cdot \hat{U}_{i,j}.$$

This assumption is very similar to what the CL reserving method assumes for claims data triangles (cumulative payments or incurred losses), and in the context of my model, the factors (g_j) can be expected to be close to 1 since the relationship is established between estimated ultimate losses.

Based on the available ultimate estimates data, the main result for the one-year reserve risk will be

$$\widehat{\text{msep}}_{\sum_{i=0}^{I}\widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0)$$
$$= \sum_{i=1}^{I}\left[\hat{\sigma}_{I-i}^2 \cdot \hat{U}_{i,I-i} + (\hat{g}_{I-i}-1)^2 \cdot \hat{U}_{i,I-i}^2\right]$$
$$+ 2\sum_{1\le i<j\le I}(\hat{g}_{I-i}\cdot\hat{g}_{I-j} - \hat{g}_{I-i} - \hat{g}_{I-j} + 1)\cdot \hat{U}_{i,I-i}\cdot\hat{U}_{j,I-j}. \quad (1.1)$$

where, for a specific development period j, the estimated development factor $\hat{g}_j$ is a weighted average of the individual accident year development factors $\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}}$, while $\hat{\sigma}_j^2$ is a measure of the volatility of these development factors around their weighted average in absolute terms, i.e., scaled to the estimated ultimate amounts.

Moreover, based on the available ultimate estimates data, the main result for the total run-off reserve risk will be

$$\widehat{\text{msep}}_{\sum_{i=0}^{I} U^i|\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I} U^i \middle| \mathcal{F}_I\right]\right)$$

$$= \sum_{i=1}^{I}\left\{\begin{array}{l}\left[\sum_{k=I-i}^{I-1}\left(\prod_{j=I-i}^{k-1}\hat{g}_j\right)\cdot\hat{\sigma}_k^2\cdot\left(\prod_{j=k+1}^{I-1}\hat{g}_j^2\right)\right]\cdot\hat{U}_{i,I-i} \\ \qquad +\left(1-\prod_{j=I-i}^{I-1}\hat{g}_j\right)^2\cdot\hat{U}_{i,I-i}^2\end{array}\right\}$$

$$+2\sum_{1\le i<j\le I}\left(1-\prod_{k=I-i}^{I-1}\hat{g}_k\right)\left(1-\prod_{k=I-j}^{I-1}\hat{g}_k\right)\cdot\hat{U}_{i,I-i}\cdot\hat{U}_{j,I-j}. \tag{1.2}$$

Remarks:

- Both estimator (1.1) and estimator (1.2) are given by the sum of two components: the first component is the sum of all risk estimators for single accident years (first row); the second component is an additional covariance term (second row) as it generally appears when calculating the variance of the sum of random variables.

  Moreover, the first component of each estimator consists of the sum of two terms representing the process variance (first term) and the parameter& model uncertainty (second term), respectively.
- Note that both estimators (1.1) and (1.2) can be easily implemented in a spreadsheet, and therefore they have a high potential for application in actuarial practice.
- Within my model the following implication holds true:
  - No parameter&model uncertainty in the ultimate estimates generating process implies the proportionality factors (g_j) to be equal to 1.
  - Note that this is reflected within my results since in case of no underlying parameter&model uncertainty, the estimated factors ($\hat{g}_j$) will be forced to be equal to 1, and as a consequence, the parameter&model uncertainty and the covariance terms in estimators (1.1) and (1.2) are equal to 0.

The paper will be organized as follows:

In Section 2, I provide some technical notations and definitions.

In Section 3, I precisely formulate my model assumptions for the ultimate estimates which allows me to derive unbiased estimators for the model parameters without assuming independence between accident years.

In Section 4, I derive MSEP for the one-year reserve risk within my model.

In Section 5, I derive MSEP of the total run-off reserve risk within my model.

In Section 6, I compare my formulas with Mack's (total run-off view, 1993) and Merz and Wüthrich's (one-year view, 2008) formulas and provide some toy numerical examples for didactic purposes.

In the appendices, technically oriented readers can find all the rigorous details for deriving the presented formulas.

For practically oriented readers, throughout the paper I will evaluate step by step the following numerical example (see Table 2) for which I show here the main results.

The estimated parameters ($\hat{g}_j$) and ($\hat{\sigma}_j^2$) (see also Section 3.1 for more insights) are given by

$$\hat{g}_j = \frac{\sum_{i=0}^{I-j-1}\hat{U}_{i,j+1}}{\sum_{i=0}^{I-j-1}\hat{U}_{i,j}},$$

$$\hat{\sigma}_j^2 = \frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\hat{U}_{i,j}\left(\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}}-\hat{g}_j\right)^2,$$

and can be easily evaluated to be equal to the values shown in Table 3.

Table 2. Ultimate estimates ($\hat{U}_{i,j}$): Numerical example

i/j	0	1	2	3	4	5	6	7	8	9	10	11	12
0	233,927	225,734	224,101	223,193	222,383	223,724	223,743	223,765	223,669	223,601	223,558	223,558	223,558
1	230,176	224,714	222,707	221,871	225,215	223,977	224,027	223,933	223,907	224,004	224,036	224,036	
2	225,361	219,714	217,321	223,774	225,325	223,809	223,680	223,450	223,544	223,660	223,697		
3	211,284	215,963	218,664	217,849	215,981	214,561	214,391	214,395	214,468	214,445			
4	212,024	221,439	220,313	218,938	218,873	220,610	220,938	221,576	221,628				
5	289,863	298,147	295,716	293,527	293,758	294,427	294,713	294,673					
6	268,568	273,222	272,219	272,657	271,057	269,817	269,763						
7	261,945	275,360	274,935	274,262	273,275	271,810							
8	232,400	241,510	242,330	247,851	247,245								
9	235,521	251,022	267,514	267,747									
10	239,385	246,401	245,398										
11	257,904	259,549											
12	262,936												

Table 3. Estimated parameters ($\hat{g}_j$) and ($\hat{\sigma}_j^2$): Numerical example

j	0	1	2	3	4	5	6	7	8	9	10	11
$\hat{g}_j$	1.0188	1.0030	1.0024	0.9996	0.9984	1.0002	1.0002	1.0001	1.0001	1.0000	1.0000	1.0000
$\hat{\sigma}_j^2$	241.45	118.62	37.85	11.80	8.32	0.16	0.41	0.03	0.04	0.01	0.00	0.00

The square rooted estimators (1.1) and (1.2) as well as Mack's (1993) and Merz and Wüthrich's (2008) results are reported in Table 4.

Note that my formulas deliver similar results as Mack's (1993) and Merz and Wüthrich's (2008) formulas and that the latter have been evaluated using the related claims payments triangle ($C_{i,j}$), shown in Table 5.

2. Notations and definitions

Consider accident years $i \in \{0, \ldots, I\}$ and development periods $j \in \{0, \ldots, I\}$.

Denote with U^i the ultimate claim amount (random variable) for accident year i.

Denote with $\mathcal{F}_k$, $k \in \{0, \ldots, 2I\}$ the (unspecified) total information available to the insurance company at the end of calendar year k (as usual, I consider here the run-off situation, i.e. no information related to accident years $i > l$ is taken into account).

Denote with $\hat{U}_{i,j}$ the estimated ultimate claims amount for accident year i at development period j, i.e., the estimated ultimate claims amount for accident year i at the end of calendar year $i + j$.

Define the following sets of information:

$$\mathcal{D}^I := \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0}, \ldots, \hat{U}_{i,I-i}\right\}\right) \subseteq \mathcal{F}_I,$$

$$\mathcal{D}^{I+1} := \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0}, \ldots, \hat{U}_{i,\min(I-i+1,I)}\right\}\right) \subseteq \mathcal{F}_{I+1},$$

Table 4. One-year and total run-off prediction uncertainties: Numerical example

	One-year risk $\widehat{\mathrm{msep}}^{1/2}_{\sum_{i=0}^{I} \widehat{CDR}_i(I+1)\mid\mathcal{F}_I}(0)$	Total run-off risk $\widehat{\mathrm{msep}}^{1/2}_{\sum_{i=0}^{I} U^i\mid\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I} U^i \mid \mathcal{F}_I\right]\right)$
My model	12,025	15,228
Underlying CL model	11,203 (Merz-Wüthrich)	13,457 (Mack)

$$\mathcal{D}_j^{j+k} := \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0}, \ldots, \hat{U}_{i,\max(j+k-i,j)}\right\}\right),$$

$$k, j \in \{0, \ldots, I\}.$$

Remarks:

- As already mentioned, the total information available, $\mathcal{F}_k$, $k \in \{0, \ldots, 2I\}$, is unspecified, i.e., it could be generated by the claims payments or by the claims-incurred amounts (or even both) and could include additional information.
- The information, $\mathcal{D}^k$, $k \in \{0, \ldots, 2I\}$, is a subset of the total information $\mathcal{F}_k$ and contains the available information given by the ultimate estimates only, which means that I do not focus anymore on the underlying claims payments or claims incurred amounts.
- The unusual information, $\mathcal{D}_j^{j+k}$, $k, j \in \{0, \ldots, I\}$, contains the ultimate estimates information at the end of calendar year $j + k$ as well as the ultimate estimates information until development period j for all accident years (as introduced in Dahms 2012).

3. The model assumptions

As outlined in the Introduction, I will not specify the stochastic reserving model supporting the reserving methodology but will

- assume the underlying reserving methodology is generating ultimate estimates, and
- assume a stochastic model for the ultimate estimates.

In mathematical terms, I make the following model assumptions:

Model Assumptions 1 (ultimate estimates model)

(A)

– The "best estimate" ultimate, $E[U^i|\mathcal{F}_{i+j}]$, $i, j \in \{0, \ldots, I\}$, can be identified to be given as a

Table 5. Claims payments ($C_{i,j}$): Numerical example

i/j	0	1	2	3	4	5	6	7	8	9	10	11	12
0	157,495	212,770	219,680	220,978	221,276	223,724	223,743	223,765	223,669	223,601	223,558	223,558	223,558
1	154,969	213,352	219,201	220,469	222,751	223,958	224,005	224,030	223,975	224,048	224,036	224,036	
2	152,833	209,969	214,692	220,040	223,467	223,754	223,752	223,593	223,585	223,688	223,697		
3	144,223	207,644	212,443	214,108	214,661	214,610	214,564	214,484	214,459	214,459			
4	145,612	209,604	214,161	215,982	217,962	220,783	221,078	221,614	221,616				
5	196,695	282,621	288,676	290,036	292,206	294,531	294,671	294,705					
6	181,381	260,308	266,497	269,130	269,404	269,691	269,720						
7	177,168	263,130	268,848	270,787	271,624	271,688							
8	156,505	230,607	237,102	244,847	245,940								
9	157,839	239,723	261,213	264,755									
10	159,429	233,309	239,800										
11	169,990	246,019											
12	173,377												

function of $\mathcal{F}_{i+j}$-measurable random variables and a collection of underlying parameters.

– $\hat{U}_{i,j} = \hat{E}[U^i | \mathcal{F}_{i+j}]$, $i, j \in \{0, \ldots, I\}$ is an estimated ultimate position obtained by replacing the unknown underlying parameters with appropriately $\mathcal{F}_{i+j}$-measurable parameter estimators.

– $U^i = \hat{U}_{i,I}$, $i \in \{0, \ldots, I\}$, i.e., at development period I the ultimate estimates are fully developed.

(B)

– There exist parameters $g_0, \ldots, g_{I-1}$ and $\sigma_0^2, \ldots, \sigma_{I-1}^2$ such that for all $i \in \{0, \ldots, I\}$ and $j \in \{0, \ldots, I-1\}$,

$$E\left[\hat{U}_{i,j+1} | \mathcal{D}_j^{j+i}\right] = g_j \cdot \hat{U}_{i,j},$$

$$Var\left(\hat{U}_{i,j+1} | \mathcal{D}_j^{j+i}\right) = \sigma_j^2 \cdot \hat{U}_{i,j}.$$

Remarks:

- Assumption (A) expresses the traditional ultimate estimates representation in claims reserving (see equations 2.9 and 2.10 in Wüthrich and Merz 2008) and allows one to model the situation where a fully well-defined stochastic model supporting the reserving methodology is missing.
- Note that I do not require any quality assumptions with respect to the underlying parameter estimators. This allows for more flexibility since ad hoc estimators can be considered as well (which is very common in practice; see, e.g., the concluding remarks in Wüthrich and Merz 2008, p. 390).
- Assumption (B) was inspired by the "Linear Stochastic Reserving Method" assumption in Dahms (2012); note that no independence assumption between accident years is stated.
- The above-mentioned underlying parameters are the ones that need to be estimated when generating the ultimate estimates $\hat{U}_{i,j}$.
- Note that considering reserving methodologies for which the unconditional unbiased property related to the ultimate estimates is fulfilled (e.g., in case of no underlying parameter&model uncertainty), i.e., for which it holds true

$$E\left[\hat{U}_{i,j}\right] = E\left[U^i\right], \quad i, j \in \{0, \ldots, I\}.$$

my model assumptions imply

$$E\left[U^i\right] = E\left[\hat{U}_{i,j+1}\right] = E\left[E\left[\hat{U}_{i,j+1} | \mathcal{D}_j^{j+i}\right]\right]$$
$$= E\left[g_j \cdot \hat{U}_{i,j}\right] = g_j \cdot E\left[U^i\right].$$

As a consequence, the parameters (g_j) are equal to 1, and therefore they do not need to be estimated (see the remarks in Section 3.1).

In this respect, please note that in practice, the above unconditional unbiased property is mostly approximately fulfilled but not exactly fulfilled, even when considering well-established reserving methodologies like the traditional Bornhuetter-Ferguson (BF; see Wüthrich and Merz 2008; Mack 2008; Saluz, Gisler, and Wüthrich 2011) method (see equation 2.15 in Wüthrich and Merz 2008) or the Generalized Linear Models (see remark 6.15 in Wüthrich and Merz 2008). Therefore, the parameters (g_j) in my model are generally close to 1 but are not necessary equal to 1.

3.1. Parameter estimation

The parameters (g_j) and (σ_j^2) can be estimated by the following $\sigma(\bigcup_{i=0}^{I}\{\hat{U}_{i,0}, \ldots, \hat{U}_{i,j}\})$-conditionally unbiased estimators:

$$\hat{g}_j := \frac{\sum_{i=0}^{I-j-1} \hat{U}_{i,j+1}}{\sum_{i=0}^{I-j-1} \hat{U}_{i,j}}, \quad j \in \{0, \ldots, I-1\}, \tag{3.1}$$

$$\hat{\sigma}_j^2 := \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} \hat{U}_{i,j} \left(\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}} - \hat{g}_j \right)^2,$$
$$j \in \{0, \ldots, I-2\}. \tag{3.2}$$

For $\hat{\sigma}_{I-1}^2$ I use the conventional estimator obtained by extrapolation (see Mack 1993), given by

$$\hat{\sigma}_{I-1}^2 = \min\left\{\hat{\sigma}_{I-3}^2, \hat{\sigma}_{I-2}^2, \frac{\hat{\sigma}_{I-2}^4}{\hat{\sigma}_{I-3}^2}\right\}. \tag{3.3}$$

Proof. See Appendix A.1. □

Remarks:

- In case the unconditional unbiased property related to the ultimate estimates is fulfilled, the parameters (g_j) do not need to be estimated, and I set

$$\hat{g}_j = 1, \quad j \in \{0, \ldots, I-1\}.$$

Moreover, the parameter estimator for σ_j^2 needs to be slightly modified to

$$\hat{\sigma}_j^2 = \frac{1}{I-j} \sum_{i=0}^{I-j-1} \hat{U}_{i,j} \left(\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}} - 1 \right)^2,$$

$$j \in \{0, \ldots, I-2\}.$$

- The estimated parameters $(\hat{g}_j)$ and $(\hat{\sigma}_j^2)$ related to my numerical example are shown in Table 3.
- Note that my theory easily can be extended for considering the case wherein the accident years are not necessarily fully developed at the end of development period I. In this case tail parameters $\hat{g}_I$ and $\hat{\sigma}_I^2$ are required (e.g., obtained by additional extrapolation), and the sums in my results (1.1) and (1.2) must be extended by the additional terms where $i = 0$ or $j = 0$.

3.2. Some basic results

In this section I provide some basic results that can be derived from Model Assumptions 1.

These will play a central role in Sections 4 and 5 when deriving the estimators for the prediction uncertainties.

The following relationships hold true:

$$E\left[\hat{U}_{i,I-i+1} \middle| \mathcal{D}^I\right] = \hat{U}_{i,I-i} \cdot g_{I-i}, \quad i \geq 1, \tag{3.4}$$

$$E\left[\hat{U}_{i,I} \middle| \mathcal{D}^I\right] = \hat{U}_{i,I-i} \cdot \prod_{j=I-i}^{I-1} g_j, \quad i \geq 1, \tag{3.5}$$

$$E\left[\hat{U}_{i,I} \middle| \mathcal{D}^{I+1}\right] = \hat{U}_{i,I-i+1} \cdot \prod_{j=I-i+1}^{I-1} g_j, \quad i \geq 2. \tag{3.6}$$

Furthermore, since $\hat{U}_{i,I-i}$ is $\mathcal{D}^I$-measurable, it also holds true that

$$\begin{aligned}
&Var\left(\hat{U}_{i,I-i+1} \middle| \mathcal{D}^I\right) \\
&\quad = E\left[Var\left(\hat{U}_{i,I-i+1} \middle| \mathcal{D}_{I-i}^{I-i+i}\right) \middle| \mathcal{D}^I\right] \\
&\quad\quad + Var\left(E\left[\hat{U}_{i,I-i+1} \middle| \mathcal{D}_{I-i}^{I-i+i}\right] \middle| \mathcal{D}^I\right) \\
&\quad = E\left[\sigma_{I-i}^2 \cdot \hat{U}_{i,I-i} \middle| \mathcal{D}^I\right] + Var\left(g_{I-i} \cdot \hat{U}_{i,I-i} \middle| \mathcal{D}^I\right) \\
&\quad = \sigma_{I-i}^2 \cdot E\left[\hat{U}_{i,I-i} \middle| \mathcal{D}^I\right] + g_{I-i}^2 \cdot \underbrace{Var\left(\hat{U}_{i,I-i} \middle| \mathcal{D}^I\right)}_{=0} \\
&\quad = \sigma_{I-i}^2 \cdot \hat{U}_{i,I-i}, \quad i \geq 1.
\end{aligned} \tag{3.7}$$

Remark:

Please note that in the following sections, I will make use of the following notation:

$\hat{E}[\cdot]$ (respectively $\widehat{Var}(\cdot)$) will denote the estimator for $E[\cdot]$ (respectively $Var(\cdot)$) which is obtained by final replacement of the unknown parameters (g_j), (σ_j^2) with their parameter estimators $(\hat{g}_j)$, $(\hat{\sigma}_j^2)$ after having performed ordinary computations.

3.3. Three examples

To clarify Model Assumptions 1, part (A), in this section I provide three examples of methodologies for generating ultimate estimates. The first one is the standard CL methodology; the second one is a modification of CL methodology, which better reflects what is generally done in actuarial practice, when parameters are adjusted according to expert judgment. The third is a credibility mixture between CL and BF.

3.3.1. Standard CL methodology

I denote with $C_{i,j}$ the cumulative payments for accident year $i \in \{0, \ldots, I\}$ up to development period $j \in \{0, \ldots, I\}$ and with $\mathcal{F}_k$, $k \in \{0, \ldots, 2I\}$, the total information available to the insurance company at the end of calendar year k, which in this case is given by

$$\mathcal{F}_k = \sigma\left(\{C_{i,j}\}_{i+j \leq k}\right).$$

Under the standard CL framework, the underlying parameters are given by a collection of factors (f_j)

and the quantities $E[U^i|\mathcal{F}_{i+j}]$ and $\hat{U}_{i,j}$, respectively, are given by

$$E\left[U^i\middle|\mathcal{F}_{i+j}\right] = C_{i,j}\cdot\prod_{l=j}^{I-1} f_l,\quad i,j\in\{0,\ldots,I\},\qquad(3.8)$$

$$\hat{U}_{i,j} = C_{i,j}\cdot\prod_{l=j}^{I-1}\hat{f}_l^{(i+j)},\quad i,j\in\{0,\ldots,I\},\qquad(3.9)$$

where the estimated CL factors at the end of calendar year k are given by

$$\hat{f}_j^{(k)} := \begin{cases}\dfrac{\sum_{i=0}^{\min(I,k-j-1)} C_{i,j+1}}{\sum_{i=0}^{\min(I,k-j-1)} C_{i,j}}, & k-j\geq 1\\ \bar{f}_j, & k-j<1\end{cases},$$

$$k\in\{0,\ldots,2I\},\qquad(3.10)$$

and $\bar{f}_0,\ldots,\bar{f}_{I-1}$ are known a priori expected CL factors.

Remarks:

- The above choice of the estimated CL factors can be justified by Mack's distribution-free model assumptions (see Mack 1993).
- Note that in practice, the standard CL methodology is not often rigorously applied, since the estimated CL factors are often modified and/or smoothed according to actuarial judgment. This fact can be interpreted as one does not perfectly believe in Mack's (1993) distribution-free model.
- Note that assuming my Model Assumptions 1 with respect to the standard CL methodology (as defined above) implies that the cumulative payments process $(C_{i,j})$ fulfills conditions that do not comply with Mack's (1993) distribution-free model assumptions. As a consequence, one can expect that the uncertainty estimators derived from my model do not necessary coincide with Mack's (1993) and Merz and Wüthrich's (2008) formulas.

The implied conditions related to the cumulative payments process $(C_{i,j})$ are similar but more complex than in Mack's (1993) distribution-free model.

3.3.2. Modified CL methodology

In this section I define a modified CL methodology that better reflects common actuarial practice.

I consider the underlying filtration given by

$$\mathcal{F}_k = \sigma\left(\{C_{i,j},\varepsilon_{i,j}\}_{i+j\leq k}\right),$$

where the variables $\varepsilon_{i,j}$ are intended to model an adjustment to the standard CL estimator in order to take into account additional information not included in the information generated by the claims payments.

The underlying parameters are given by a collection of factors (f_j), and the quantities $E[U^i|\mathcal{F}_{i+j}]$ and $\hat{U}_{i,j}$, respectively, are given by

$$E\left[U^i\middle|\mathcal{F}_{i+j}\right] = \varepsilon_{i,j}\cdot C_{i,j}\cdot\prod_{l=j}^{I-1} f_l,\quad i,j\in\{0,\ldots,I\},\qquad(3.11)$$

$$\hat{U}_{i,j} = \varepsilon_{i,j}\cdot C_{i,j}\cdot\prod_{l=j}^{I-1}\tilde{f}_l^{(i+j)},\quad i,j\in\{0,\ldots,I\},\qquad(3.12)$$

where $(\tilde{f}_l^{(i+j)})$ are appropriate $\mathcal{F}_{i+j}$-measurable estimators for (f_l).

Remarks:

- The above estimated CL factors $(\tilde{f}_l^{(k)})$ at the end of calendar year k could, for instance, coincide with the—in this case ad hoc—standard CL estimators $(\hat{f}_l^{(k)})$.
- Compared to the standard CL methodology, the modified CL methodology better reflects the usual actuarial behavior observed in practice, where the standard CL ultimate estimators/factors are possibly adjusted by hand, using expert opinion to take into account additional information such as change of legal practice, high inflation, and job market information.

3.3.3. Credibility mixture between CL and BF

In this section I define a credibility mixture methodology between CL and BF following the setup of

Section 4.2 in Wüthrich and Merz (2008). I consider the underlying filtration given by

$$\mathcal{F}_k = \sigma\left(\{C_{i,j}\}_{i+j\le k}, \{\hat{\mu}_i\}_{i\le k}\right),$$

where $(\hat{\mu}_i)$ are unbiased estimators for $\mu_i = E[C_{i,I}]$, which are independent from the claims payments variables.

The underlying parameters are given by a collection of factors (f_j), (μ_i), $(c_{i,j})$ and the quantity $\hat{U}_{i,j}$ is given by

$$\hat{U}_{i,j} = \hat{c}_{i,j}\cdot\underbrace{C_{i,j}\cdot\prod_{l=j}^{I-1}\hat{f}_l^{(i+j)}}_{\text{CL component}} + (1-\hat{c}_{i,j})$$
$$\cdot\underbrace{\left(C_{i,j}+\left(1-\frac{1}{\prod_{l=j}^{I-1}\hat{f}_l^{(i+j)}}\right)\hat{\mu}_i\right)}_{\text{BF component}},\quad i,j\in\{0,\ldots,I\}. \tag{3.13}$$

where $\hat{c}_{i,j}$ is an appropriate estimator for $c_{i,j}$, for instance, the one obtained by minimizing the appropriate unconditional MSEP for known parameters (f_j) (see remarks 4.12 in Wüthrich and Merz (2008)).

Moreover, the quantity $E[U^i|\mathcal{F}_{i+j}]$ can be given by

$$E\left[U^i\middle|\mathcal{F}_{i+j}\right] = c_{i,j}\cdot C_{i,j}\cdot\prod_{l=j}^{I-1} f_l + (1-c_{i,j})$$
$$\cdot\left(C_{i,j}+\left(1-\frac{1}{\prod_{l=j}^{I-1} f_l}\right)\mu_i\right),$$
$$i,j\in\{0,\ldots,I\}. \tag{3.14}$$

Remark:

The above ultimate estimates $(\hat{U}_{i,j})$ are motivated by the underlying stochastic model given by assumption 4.11 in Wüthrich and Merz (2008), but note that within that model the standard CL estimators $(\hat{f}_l^{(k)})$ are ad hoc parameter estimators and that one cannot explicitly prove (3.14) to hold true.

3.4. Concluding remark

Finishing this section I would like to recall and highlight again that my methodology has been appositely designed for modeling the situation where a fully well-defined stochastic model supporting the reserving methodology is missing (e.g., due to mixing of reserving methods or when underlying parameter estimators are adjusted by hand, as is usually done in practice).

In this case I am free to assume my Model Assumptions 1 to describe the evolution of the ultimate estimates.

Therefore, if a stochastic reserving model supporting a reserving methodology can be fully well defined, as for the standard CL methodology where the underlying parameter estimators can be perfectly motivated and model validation delivers good results, then I believe it would be better to use (if known) the uncertainty estimators derived within the underlying model itself and not follow my theory.

However, since

- underlying stochastic reserving models often do rely on restrictive assumptions,
- the unbiasedness property of the parameter estimators cannot always be exactly proved, and
- the prediction uncertainty estimators are sometimes derived using approximations, in this case one should not forget to allow also for a (model) risk loading.

As a consequence, since the latter is generally not easily quantified, I consider my theory and the related formulas to be in any case a very valuable alternative.

4. One-year prediction uncertainty

In this section, I concentrate on the one-year reserve risk. My goal is to derive the estimator (1.1) for MSEP of the one-year prediction uncertainty. As usual in claim reserving, I first focus on the result for a single accident year and then, in a second step, derive an estimator for the aggregated view.

Let us first define the observed claims development result $\widehat{\text{CDR}}_i(I+1)$ as follows:

$$\widehat{\text{CDR}}_i(I+1) := \hat{E}\left[U^i\middle|\mathcal{F}_{I+1}\right] - \hat{E}\left[U^i\middle|\mathcal{F}_I\right],\quad i\ge 1.$$

4.1. Single accident year

For a single accident year $i \geq 1$, MSEP of the one-year prediction uncertainty is given by

$$\begin{aligned}&\text{msep}_{\widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0)\\&:= E\left[\left(\hat{E}\left[U^i|\mathcal{F}_{I+1}\right]-\hat{E}\left[U^i|\mathcal{F}_I\right]-0\right)^2\Big|\mathcal{F}_I\right]\\&=\left[\left(\hat{U}_{i,I-i+1}-\hat{U}_{i,I-i}\right)^2\Big|\mathcal{F}_I\right].\end{aligned} \tag{4.1}$$

4.1.1. One-year uncertainty estimator

Making use of the relationships derived in Section 3.2, I can derive the following estimator (see Appendix B.1 for more details):

Estimator 1 (one-year reserve risk estimator for single accident year). *Under Model Assumptions 1 I have the following estimator for the one-year prediction uncertainty for a single accident year:*

$$\widehat{\text{msep}}_{\widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0)=\hat{\sigma}^2_{I-i}\cdot\hat{U}_{i,I-i}+(\hat{g}_{I-i}-1)^2\cdot\hat{U}^2_{i,I-i},\quad i\geq 1. \tag{4.2}$$

Remarks:

- Note that since

$$\begin{aligned}&E\left[\left(\hat{U}_{i,I-i+1}-\hat{U}_{i,I-i}\right)^2\Big|\mathcal{D}^I\right]\\&=E\left[E\left[\left(\hat{U}_{i,I-i+1}-\hat{U}_{i,I-i}\right)^2\Big|\mathcal{F}_I\right]\Big|\mathcal{D}^I\right],\quad i\geq 1,\end{aligned}$$

  my one-year reserve risk estimator fulfills the following unbiasedness property:

$$\widehat{\text{msep}}_{\widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0)=\hat{E}\left[\text{msep}_{\widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0)|\mathcal{D}^I\right],\quad i\geq 1. \tag{4.3}$$

- The one-year reserve risk estimators for single accident year related to my numerical example are shown in Table 6.

Table 6. One-year prediction uncertainty for single accident year: Numerical example

i	$\hat{U}_{i,I-i}$	$\widehat{\text{msep}}_{\widehat{\text{CDR}}_i(I+1)\|\mathcal{F}_I}(0)$	$\widehat{\text{msep}}^{1/2}_{\widehat{\text{CDR}}_i(I+1)\|\mathcal{F}_I}(0)$
0	223,558	0	0
1	224,036	0	0
2	223,697	0	0
3	214,445	1,977	44
4	221,628	8,941	95
5	294,673	8,995	95
6	269,763	114,752	339
7	271,810	45,029	212
8	247,245	2,214,687	1,488
9	267,747	3,170,237	1,781
10	245,398	9,631,068	3,103
11	259,549	31,380,299	5,602
12	262,936	87,858,844	9,373

4.1.2. Artificial split of the one-year prediction uncertainty between process variance and parameter&model uncertainty

Note that MSEP of the one-year prediction uncertainty can be split into three components (process variance, parameter&model uncertainty, and mixed term) as follows (see Appendix B.2 for more details):

$$\begin{aligned}&\text{msep}_{\widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0)\\&:= E\left[\left(\hat{E}\left[U^i|\mathcal{F}_{I+1}\right]-\hat{E}\left[U^i|\mathcal{F}_I\right]-0\right)^2\Big|\mathcal{F}_I\right]\\&=\underbrace{Var\left(E\left[U^i|\mathcal{F}_{I+1}\right]\Big|\mathcal{F}_I\right)}_{\text{process variance}}\\&+\underbrace{\begin{matrix}E\left[\left(\hat{U}_{i,I-i+1}-E\left[U^i|\mathcal{F}_{I+1}\right]^2\right)\Big|\mathcal{F}_I\right]\\+E\left[\left(\hat{U}_{i,I-i}-E\left[U^i|\mathcal{F}_I\right]^2\right)\Big|\mathcal{F}_I\right]\end{matrix}}_{\text{parameter\&model uncertainty: first term}}\\&\underbrace{-2E\left[\begin{matrix}\left(\hat{U}_{i,I-i}-E\left[U^i|\mathcal{F}_I\right]\right)\\\cdot\left(\hat{U}_{i,I-i+1}-E\left[U^i|\mathcal{F}_{I+1}\right]\right)\end{matrix}\Bigg|\mathcal{F}_I\right]}_{\text{parameter\&model uncertainty: second term}}\\&\underbrace{+2E\left[\begin{matrix}\left(\hat{U}_{i,I-i+1}-E\left[U^i|\mathcal{F}_{I+1}\right]\right)\\\cdot\left(E\left[U^i|\mathcal{F}_{I+1}\right]-E\left[U^i|\mathcal{F}_I\right]\right)\end{matrix}\Bigg|\mathcal{F}_I\right]}_{\text{mixed term}}.\end{aligned} \tag{4.4}$$

In Appendices B.3, B.4, and B.5, I will additionally derive an estimator for each of the above terms.

As just shown, I cannot clearly split the one-year prediction uncertainty between process variance and parameter&model uncertainty due to the presence of the mixed term.

Nevertheless, I can at least provide the following artificial split of the one-year prediction uncertainty estimator:

$$\widehat{\text{msep}}_{\widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0) = \underbrace{\hat{\sigma}^2_{I-i} \cdot \hat{U}_{i,I-i}}_{\text{“estimated process variance”}} + \underbrace{(\hat{g}_{I-i} - 1)^2 \cdot \hat{U}^2_{i,I-i}}_{\text{“estimated parameter\&model uncertainty”}}, \quad i \geq 1,$$

since conditional on the true underlying model and for known parameters (i.e., no underlying parameter&model uncertainty) the estimated parameters $\hat{g}_j$, $j \in \{0, \ldots, I-1\}$ are forced to be equal to 1, and in consequence the above estimated parameter&model uncertainty term vanishes.

4.2. Aggregated accident years

In this section, I take care of the aggregated accident year view. For aggregated accident years, MSEP of the one-year prediction uncertainty is given by

$$\begin{aligned}
&\text{msep}_{\sum_{i=0}^{I} \widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0) \\
&:= E\left[\left(\underbrace{\sum_{i=0}^{I}\left(\hat{E}[U^i|\mathcal{F}_{I+1}] - \hat{E}[U^i|\mathcal{F}_I]\right)}_{=:\widehat{\text{CDR}}_i(I+1)} - 0\right)^2 \middle| \mathcal{F}_I\right] \\
&= \sum_{i=1}^{I} E\left[\left(\hat{E}[U^i|\mathcal{F}_{I+1}] - \hat{E}[U^i|\mathcal{F}_I]\right)^2 \middle| \mathcal{F}_I\right] \\
&\quad + 2\sum_{1\leq i<j\leq I} E\left[\begin{matrix}\left(\hat{E}[U^i|\mathcal{F}_{I+1}] - \hat{E}[U^i|\mathcal{F}_I]\right) \\ \cdot\left(\hat{E}[U^j|\mathcal{F}_{I+1}] - \hat{E}[U^j|\mathcal{F}_I]\right)\end{matrix} \middle| \mathcal{F}_I\right].
\end{aligned} \tag{4.5}$$

For the above first term $\sum_{i=1}^{I} E[(\hat{E}[U^i|\mathcal{F}_{I+1}] - \hat{E}[U^i|\mathcal{F}_I])^2|\mathcal{F}_I]$, I already know an appropriate estimator since it refers to the one-year prediction uncertainty for single accident years. Therefore it remains only to estimate the second term, which I will call the one-year covariance term. This is done in the next subsection.

4.2.1. One-year uncertainty estimator (for aggregated accident years)

I first state the following lemma:

Lemma 1. *Under Model Assumptions 1 and for $i < j$ it holds true that*

$$Cov\left(\hat{U}_{i,I-i+1}, \hat{U}_{j,I-j+1} \middle| \mathcal{D}^I\right) = 0. \tag{4.6}$$

Proof. See Appendix B.6. □

Using Lemma 1, I obtain the following estimator for the one-year covariance term (see Appendix B.7 for more details):

$$\begin{aligned}
&2\sum_{1\leq i<j\leq I} \hat{E}\left[\left(\hat{U}_{i,I-i+1} - \hat{U}_{i,I-i}\right)\left(\hat{U}_{j,I-j+1} - \hat{U}_{j,I-j}\right)\middle|\mathcal{D}^I\right] \\
&= 2\sum_{1\leq i<j\leq I} (\hat{g}_{I-i} \cdot \hat{g}_{I-j} - \hat{g}_{I-i} - \hat{g}_{I-j} + 1) \cdot U_{i,I-i} \cdot U_{j,I-j}.
\end{aligned} \tag{4.7}$$

Remark:

The one-year covariance term estimator related to my numerical example is given by

$$\begin{aligned}
&2\sum_{1\leq i<j\leq I} \hat{E}\left[\left(\hat{U}_{i,I-i+1} - \hat{U}_{i,I-i}\right)\left(\hat{U}_{j,I-j+1} - \hat{U}_{j,I-j}\right)\middle|\mathcal{D}^I\right] \\
&= 10{,}167{,}783.
\end{aligned}$$

As a consequence I can derive estimator (1.1).

Estimator 2 (one-year reserve risk estimator for aggregated accident years). *Under Model Assumptions 1 I have the following estimator for the one-year prediction uncertainty for aggregated accident years:*

$$\begin{aligned}
&\widehat{\text{msep}}_{\sum_{i=0}^{I} \widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0) \\
&= \sum_{i=1}^{I}\left[\hat{\sigma}^2_{I-i} \cdot \hat{U}_{i,I-i} + (\hat{g}_{I-i} - 1)^2 \cdot \hat{U}^2_{i,I-i}\right] \\
&\quad + 2\sum_{1\leq i<j\leq I} (\hat{g}_{I-i} \cdot \hat{g}_{I-j} - \hat{g}_{I-i} - \hat{g}_{I-j} + 1) \cdot \hat{U}_{i,I-i} \cdot \hat{U}_{j,I-j}.
\end{aligned}$$

Table 7. One-year prediction uncertainties: Numerical example

i	$\hat{U}_{i,I-i}$	$\widehat{\text{msep}}_{\widehat{\text{CDR}}_i(I+1)\vert\mathcal{F}_I}(0)$	$\widehat{\text{msep}}^{1/2}_{\widehat{\text{CDR}}_i(I+1)\vert\mathcal{F}_I}(0)$
0	223,558	0	0
1	224,036	0	0
2	223,697	0	0
3	214,445	1,977	44
4	221,628	8,941	95
5	294,673	8,995	95
6	269,763	114,752	339
7	271,810	45,029	212
8	247,245	2,214,687	1,488
9	267,747	3,170,237	1,781
10	245,398	9,631,068	3,103
11	259,549	31,380,299	5,602
12	262,936	87,858,844	9,373
covariance term		10,167,783	
	$\sum_{i=0}^{I}\hat{U}_{i,I-i}$	$\widehat{\text{msep}}_{\sum_{i=0}^{I}\widehat{\text{CDR}}_i(I+1)\vert\mathcal{F}_I}(0)$	$\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}\widehat{\text{CDR}}_i(I+1)\vert\mathcal{F}_I}(0)$
Total	3,226,487	144,602,611	12,025

Remarks:

- It is straightforward that the above estimator does fulfill the following unbiasedness property:

$$\widehat{\text{msep}}_{\sum_{i=0}^{I}\widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0)=\hat{E}\left[\text{msep}_{\sum_{i=0}^{I}\widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0)\middle|\mathcal{D}^I\right]. \tag{4.8}$$

- The one-year reserve risk estimator for aggregated accident years related to my numerical example is shown in Table 7.

5. Total run-off prediction uncertainty

In this section I concentrate on the total run-off reserve risk. My goal is to derive the estimator (1.2) for MSEP of the total run-off prediction uncertainty. As usual I first focus on the result for a single accident year and then, in a second step, derive an estimator for the aggregated view.

5.1. Single accident year

For a single accident year $i \geq 1$, MSEP of the total run-off prediction uncertainty is given by

$$\begin{aligned}
&\text{msep}_{U^i|\mathcal{F}_I}\left(\hat{E}\left[U^i|\mathcal{F}_I\right]\right)\\
&:=E\left[\left(U^i-\hat{E}\left[U^i|\mathcal{F}_I\right]\right)^2\middle|\mathcal{F}_I\right]\\
&=E\left[\left(U^i-E\left[U^i|\mathcal{F}_I\right]\right)^2\middle|\mathcal{F}_I\right]\\
&\quad+E\left[\left(\hat{E}\left[U^i|\mathcal{F}_I\right]-E\left[U^i|\mathcal{F}_I\right]\right)^2\middle|\mathcal{F}_I\right]\\
&\quad+2\underbrace{E\left[\begin{array}{c}\left(U^i-E\left[U^i|\mathcal{F}_I\right]\right)\\ \cdot\left(E\left[U^i|\mathcal{F}_I\right]-\hat{E}\left[U^i|\mathcal{F}_I\right]\right)\end{array}\middle|\mathcal{F}_I\right]}_{=0}\\
&=\underbrace{E\left[\left(\hat{U}_{i,I}-E\left[\hat{U}_{i,I}|\mathcal{F}_I\right]\right)^2\middle|\mathcal{F}_I\right]}_{\text{process variance}}\\
&\quad+\underbrace{E\left[\left(\hat{U}_{i,I-i}-E\left[\hat{U}_{i,I}|\mathcal{F}_I\right]\right)^2\middle|\mathcal{F}_I\right]}_{\text{parameter\&model uncertainty}}.
\end{aligned} \tag{5.1}$$

Remark: Note that the total run-off prediction uncertainty can be clearly split (unlike the one-year prediction uncertainty) between process variance and parameter&model uncertainty term.

In the next subsections I derive an estimator for both the above terms: process variance and parameter&model uncertainty. I will again make use of the relationships derived in Section 3.2.

5.1.1. Process variance estimator

I estimate the process variance term

$$E\left[\left(\hat{U}_{i,I} - E\left[\hat{U}_{i,I} \middle| \mathcal{F}_I\right]\right)^2 \middle| \mathcal{F}_I\right]$$

by

$$\hat{E}\left[\left(\hat{U}_{i,I} - E\left[\hat{U}_{i,I} \middle| \mathcal{D}^I\right]\right)^2 \middle| \mathcal{D}^I\right]$$

$$= \left[\sum_{k=I-i}^{I-1} \left(\prod_{j=I-i}^{k-1} \hat{g}_j\right) \cdot \hat{\sigma}_k^2 \cdot \left(\prod_{j=k+1}^{I-1} \hat{g}_j^2\right)\right] \cdot \hat{U}_{i,I-i}, \quad i \geq 1, \tag{5.2}$$

and I refer to Appendix C.1 for the rigorous derivation of this result.

Remark:

The process variance estimator for single accident year related to the numerical example is shown in Table 8.

Table 8. Process variance estimator for single accident year: Numerical example

i	$\hat{U}_{i,I-i}$	$\hat{E}[(\hat{U}_{i,I} - E[\hat{U}_{i,I} \vert \mathcal{D}^I])^2 \vert \mathcal{D}^I]$	$\hat{E}[(\hat{U}_{i,I} - E[\hat{U}_{i,I} \vert \mathcal{D}^I])^2 \vert \mathcal{D}^I]^{1/2}$
0	223,558	0	0
1	224,036	0	0
2	223,697	0	0
3	214,445	1,911	44
4	221,628	9,969	100
5	294,673	21,586	147
6	269,763	131,237	362
7	271,810	174,441	418
8	247,245	2,217,465	1,489
9	267,747	5,555,024	2,357
10	245,398	14,368,781	3,791
11	259,549	46,095,893	6,789
12	262,936	111,575,746	10,563

5.1.2. Parameter&model uncertainty estimator

I estimate the parameter&model uncertainty term

$$E\left[\left(\hat{U}_{i,I-i} - E\left[\hat{U}_{i,I} \middle| \mathcal{F}_I\right]\right)^2 \middle| \mathcal{F}_I\right]$$

by

$$\hat{E}\left[\left(\hat{U}_{i,I-i} - E\left[\hat{U}_{i,I} \middle| \mathcal{D}^I\right]\right)^2 \middle| \mathcal{D}^I\right] = \left(1 - \prod_{j=I-i}^{I-1} \hat{g}_j\right)^2 \cdot \hat{U}_{i,I-i}^2,$$

$$i \geq 1, \tag{5.3}$$

and I refer to Appendix C.2 for the rigorous derivation of this result.

Remark:

The parameter&model uncertainty estimator for single accident year related to the numerical example is shown in Table 9.

5.1.3. Total run-off prediction uncertainty

Combining the above results leads to the following estimator for total run-off risk for a single accident year:

Table 9. Parameter&model uncertainty estimator for single accident year: Numerical example

i	$\hat{U}_{i,I-i}$	$\hat{E}[(\hat{U}_{i,I-i} - E[\hat{U}_{i,I} \vert \mathcal{D}^I])^2 \vert \mathcal{D}^I]$	$\hat{E}[(\hat{U}_{i,I-i} - E[\hat{U}_{i,I} \vert \mathcal{D}^I])^2 \vert \mathcal{D}^I]^{1/2}$
0	223,558	0	0
1	224,036	0	0
2	223,697	0	0
3	214,445	65	8
4	221,628	1,532	39
5	294,673	6,068	78
6	269,763	16,670	129
7	271,810	33,804	184
8	247,245	53,386	231
9	267,747	121,736	349
10	245,398	69,642	264
11	259,549	1,102,668	1,050
12	262,936	36,246,911	6,021

Estimator 3 (total run-off reserve risk estimator for single accident year). *Under Model Assumptions 1 I have the following estimator for the total run-off prediction uncertainty for a single accident year:*

$$\widehat{\text{msep}}_{U^i|\mathcal{F}_I}\left(\hat{E}\left[U^i|\mathcal{F}_I\right]\right)$$

$$= \underbrace{\left[\sum_{k=I-i}^{I-1}\left(\prod_{j=I-i}^{k-1}\hat{g}_j\right)\cdot\hat{\sigma}_k^2\cdot\left(\prod_{j=k+i}^{I-1}\hat{g}_j^2\right)\right]\cdot\hat{U}_{i,I-i}}_{\text{estimated process variance}}$$

$$+\underbrace{\left(1-\prod_{j=I-i}^{I-1}\hat{g}_j\right)^2\cdot\hat{U}_{i,I-i}^2}_{\text{estimated parameter\&model uncertainty}}, \quad i\geq 1. \qquad (5.4)$$

Remarks:

- Note that (like for the artificial one-year reserve risk split) the estimated process variance term is a linear function in $\hat{U}_{i,I-i}$, whereas the estimated parameter&model uncertainty term is a quadratic function in $\hat{U}_{i,I-i}$.
- It can be shown that the above estimator does fulfill the following unbiasedness property:

$$\widehat{\text{msep}}_{U^i|\mathcal{F}_I}\left(\hat{E}\left[U^i|\mathcal{F}_I\right]\right)=\hat{E}\left[\text{msep}_{U^i|\mathcal{F}_I}\left(\hat{E}\left[U^i|\mathcal{F}_I\right]\right)\Big|\mathcal{D}^I\right],$$

$$i\geq 1. \qquad (5.5)$$

- The total run-off reserve risk estimator for single accident year related to the numerical example is shown in Table 10.

5.2. Aggregated accident years

I now take care of the aggregated accident year view. For aggregated accident years MSEP of the total run-off prediction uncertainty is given by

$$\text{msep}_{\sum_{i=0}^I U^i|\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^I U^i|\mathcal{F}_I\right]\right)$$

$$:= E\left[\left(\sum_{i=0}^I U^i-\hat{E}\left[\sum_{i=0}^I U^i|\mathcal{F}_I\right]\right)^2\Bigg|\mathcal{F}_I\right]$$

Table 10. Total run-off prediction uncertainty by accident year: Numerical example

i	$\hat{U}_{i,I-i}$	$\widehat{\text{msep}}_{U^i\vert\mathcal{F}_I}(\hat{E}[U^i\vert\mathcal{F}_I])$	$\widehat{\text{msep}}_{U^i\vert\mathcal{F}_I}(\hat{E}[U^i\vert\mathcal{F}_I])^{1/2}$
0	223,558	0	0
1	224,036	0	0
2	223,697	0	0
3	214,445	1,977	44
4	221,628	11,502	107
5	294,673	27,654	166
6	269,763	147,907	385
7	271,810	208,245	456
8	247,245	2,270,850	1,507
9	267,747	5,676,760	2,383
10	245,398	14,438,423	3,800
11	259,549	47,198,561	6,870
12	262,936	147,822,657	12,158

$$= E\left[\left(\sum_{i=0}^I\left(U^i-\hat{E}\left[U^i|\mathcal{F}_I\right]\right)\right)^2\Bigg|\mathcal{F}_I\right]$$

$$=\sum_{i=1}^I E\left[\left(U^i-\hat{E}\left[U^i|\mathcal{F}_I\right]\right)^2\Big|\mathcal{F}_I\right]$$

$$+2\sum_{1\leq i\leq j\leq I}E\left[\begin{array}{c}\left(U^i-\hat{E}\left[U^i|\mathcal{F}_I\right]\right)\\ \cdot\left(U^j-\hat{E}\left[U^j|\mathcal{F}_I\right]\right)\end{array}\Bigg|\mathcal{F}_I\right]$$

$$=\sum_{i=1}^I E\left[\left(U^i-\hat{E}\left[U^i|\mathcal{F}_I\right]\right)^2\Big|\mathcal{F}_I\right]$$

$$+2\sum_{1\leq i<j\leq I}E\left[\left(\hat{U}_{i,I}-\hat{U}_{i,I-i}\right)\left(\hat{U}_{j,I}-\hat{U}_{j,I-j}\right)\Big|\mathcal{F}_I\right]. \qquad (5.6)$$

For the above first term $\sum_{i=1}^I E[(U^i-\hat{E}[U^i|\mathcal{F}_I])^2\,|\,\mathcal{F}_I]$, I already know an appropriate estimator since it refers to the total run-off prediction uncertainty for single accident years. Therefore it remains to estimate the second term, which I will call the total run-off covariance term. This is done in the next subsection.

5.2.1. Total run-off uncertainty estimator (for aggregated accident years)

I first state the following lemma:

Lemma 2 *Under Model Assumptions 1 and for $i < j$ it holds true that*

$$Cov\left(\hat{U}_{i,I}, \hat{U}_{j,I} \middle| \mathcal{D}^I\right) = 0. \tag{5.7}$$

Proof. See Appendix C.3. □

Using Lemma 2, I obtain the following estimator for the total run-off covariance term (see Appendix C.4 for more details):

$$2 \sum_{1 \le i < j \le I} \hat{E}\left[\left(\hat{U}_{i,I} - \hat{U}_{i,I-i}\right)\left(\hat{U}_{j,I} - \hat{U}_{j,I-j}\right) \middle| \mathcal{D}^I\right]$$

$$= 2 \sum_{1 \le i < j \le I} \left(1 - \prod_{k=I-i}^{I-1} \hat{g}_k\right)\left(1 - \prod_{k=I-j}^{I-1} \hat{g}_k\right) \cdot \hat{U}_{i,I-i} \cdot \hat{U}_{j,I-j}. \tag{5.8}$$

Remark:

The total run-off covariance term estimator related to the numerical example is given by

$$2 \sum_{1 \le i < j \le I} \hat{E}\left[\left(\hat{U}_{i,I} - \hat{U}_{i,I-i}\right)\left(\hat{U}_{j,I} - \hat{U}_{j,I-j}\right) \middle| \mathcal{D}^I\right]$$

$$= 14,082,024.$$

As a consequence I can derive estimator (1.2).

Estimator 4 (total run-off reserve risk estimator for aggregated accident years). *Under Model Assumptions 1 I have the following estimator for the total run-off prediction uncertainty for aggregated accident years*

$$\widehat{\text{msep}}_{\sum_{i=0}^{I} U^i | \mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{1} U^i \middle| \mathcal{F}_I\right]\right)$$

$$= \sum_{i=1}^{I} \left\{ \begin{array}{l} \left[\sum_{k=I-i}^{I-1} \left(\prod_{j=I-i}^{k-1} \hat{g}_j\right) \cdot \hat{\sigma}_k^2 \cdot \left(\prod_{j=k+1}^{I-1} \hat{g}_j^2\right)\right] \cdot \hat{U}_{i,I-i} \\ + \left(1 - \prod_{j=I-i}^{I-1} \hat{g}_j\right)^2 \cdot \hat{U}_{i,I-i}^2 \end{array} \right\}$$

$$+ 2 \sum_{1 \le i < j \le I} \left(1 - \prod_{k=I-i}^{I-1} \hat{g}_k\right)\left(1 - \prod_{k=I-j}^{I-1} \hat{g}_k\right)$$

$$\cdot \hat{U}_{i,I-i} \cdot \hat{U}_{j,I-j}.$$

Remarks:

- It can be shown that the above estimator does fulfill the following unbiasedness property:

$$\widehat{\text{msep}}_{\sum_{i=0}^{I} U^i | \mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{1} U^i \middle| \mathcal{F}_I\right]\right)$$

$$= \hat{E}\left[\text{msep}_{\sum_{i=0}^{I} U^i | \mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{1} U^i \middle| \mathcal{F}_I\right]\right) \middle| \mathcal{D}^I\right]. \tag{5.9}$$

- The total run-off reserve risk estimator for aggregated accident years related to the numerical example is shown in Table 11.

6. Comparison with the Mack and Merz-Wüthrich formulas

In this section I compare my formulas with the Mack (total run-off view, 1993) and Merz-Wüthrich (one-year view, 2008) formulas, which are often used as benchmarks in the insurance industry. Since the Mack and the Merz-Wüthrich formulas are based on the standard CL methodology supported by Mack's distribution-free model, I also consider this setup throughout this section.

Let us recall that I denote with $C_{i,j}$ the cumulative payments for accident year $i \in \{0, \ldots, I\}$ up to development period $j \in \{0, \ldots, I\}$ and with $\mathcal{F}_k$, $k \in \{0, \ldots, 2I\}$, the total information available to the insurance company at the end of calendar year k, which is given by

$$\mathcal{F}_k = \sigma\left(\{C_{i,j}\}_{i+j \le k}\right).$$

Under the standard CL framework I define the ultimate estimates $\hat{U}_{i,j} = \hat{E}[U^i \mid \mathcal{F}_{i+j}]$ as

$$\hat{U}_{i,j} = C_{i,j} \cdot \prod_{l=j}^{I-1} \hat{f}_l^{(i+j)}, \quad i, j \in \{0, \ldots, I\},$$

Table 11. Total run-off prediction uncertainties: Numerical example

i	$\hat{U}_{i,I-i}$	$\widehat{\text{msep}}_{U^i\mid\mathcal{F}_I}(\hat{E}[U^i\mid\mathcal{F}_I])$	$\widehat{\text{msep}}_{U^i\mid\mathcal{F}_I}(\hat{E}[U^i\mid\mathcal{F}_I])^{1/2}$
0	223,558	0	0
1	224,036	0	0
2	223,697	0	0
3	214,445	1,977	44
4	221,628	11,502	107
5	294,673	27,654	166
6	269,763	147,907	385
7	271,810	208,245	456
8	247,245	2,270,850	1,507
9	267,747	5,676,760	2,383
10	245,398	14,438,423	3,800
11	259,549	47,198,561	6,870
12	262,936	147,822,657	12,158
covariance term		14,082,024	
	$\sum_{i=0}^{I}\hat{U}_{i,I-i}$	$\widehat{\text{msep}}_{\sum_{i=0}^{I}U^i\mid\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I}U^i\middle\vert\mathcal{F}_I\right]\right)$	$\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}U^i\mid\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I}U^i\middle\vert\mathcal{F}_I\right]\right)$
Total	3,226,487	231,886,560	15,228

where the estimated CL factors at the end of calendar year k are given by

$$\hat{f}_j^{(k)} = \begin{cases} \dfrac{\sum_{i=0}^{\min(I,k-j-1)} C_{i,j+1}}{\sum_{i=0}^{\min(I,k-j-1)} C_{i,j}}, & k-j \geq 1 \\ \bar{f}_j, & k-j < 1 \end{cases},$$

$$k \in \{0, \ldots, 2I\},$$

and $\bar{f}_0, \ldots, \bar{f}_{I-1}$ are known a priori expected CL factors.

6.1. Approximation results under additional conditions

Keeping in mind the remarks in Section 3.3.1, in this section I would like to establish under which conditions Mack's (1993) and Merz-Wüthrich's (2008) formulas can be approximated by my formulas.

In other words, the goal of this section is to show that, even if my Model Assumptions 1 do not combine with Mack's (1993) model assumptions, under appropriate conditions my formulas deliver results similar to Mack's (1993) and Merz-Wüthrich's (2008) formulas.

First recall that within Mack's (1993) distribution-free CL model, the variance parameters are estimated by

$$\hat{s}_j^2 := \frac{1}{I-j-1}\sum_{i=0}^{I-j-1} C_{i,j}\left(\frac{C_{i,j+1}}{C_{i,j}} - \hat{f}_j^{(I)}\right)^2,$$

$$j \in \{0, \ldots, I-2\}. \tag{6.1}$$

Moreover, for $\hat{s}_{I-1}^2$ I use the estimator proposed by Mack (1993), given by

$$\hat{s}_{I-1}^2 = \min\left\{\hat{s}_{I-3}^2, \hat{s}_{I-2}^2, \frac{\hat{s}_{I-2}^4}{\hat{s}_{I-3}^2}\right\}. \tag{6.2}$$

I will be able to prove the following result:

Approximation Result 1. *Assume Mack's (1993) distribution-free CL model as well as*

1) no parameter estimation uncertainty;
2) $\hat{f}_j^{(k)} \approx \bar{f}_j$, $\forall j, k$; and

3) $\hat{\sigma}_{I-1}^2 \approx \bar{f}_{I-1} \cdot \frac{\hat{s}_{I-1}^2}{(\bar{f}_{I-1})^2}$,

where $\bar{f}_0, \ldots, \bar{f}_{I-1}$ are known, a priori expected CL factors and $\hat{f}_j^{(k)}$ is defined as above.

Then the following approximations hold true

$$\text{msep}^{\text{Mack}}_{\sum_{i=0}^{I} U^i|\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I} U^i \middle| \mathcal{F}_I\right]\right) \approx \widehat{\text{msep}}_{\sum_{i=0}^{I} U^i|\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I} U^i \middle| \mathcal{F}_I\right]\right), \tag{6.3}$$

$$\text{msep}^{\text{MW}}_{\sum_{i=0}^{I} \widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0) \approx \widehat{\text{msep}}_{\sum_{i=0}^{I} \widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0). \tag{6.4}$$

Proof. From the additional assumptions 1) and 2) I have

$$\hat{g}_j \overset{1)}{=} \frac{\sum_{i=0}^{I-j-1} C_{i,j+1} \cdot \prod_{l=j+1}^{I-1} \bar{f}_l}{\sum_{i=0}^{I-j-1} C_{i,j} \cdot \prod_{l=j}^{I-1} \bar{f}_l} = \frac{1}{\bar{f}_j} \frac{\sum_{i=0}^{I-j-1} C_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}} \overset{2)}{\approx} 1, \quad j \in \{0, \ldots, I-1\}. \tag{6.5}$$

As a consequence, for the one-year risk it holds true that

$$\begin{aligned}
&\widehat{\text{msep}}_{\sum_{i=0}^{I} \widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0) \\
&\overset{(6.5)}{\approx} \sum_{i=1}^{I} \hat{\sigma}_{I-i}^2 \cdot \hat{U}_{i,I-i} \overset{1)}{=} \sum_{i=1}^{I} \left(\hat{U}_{i,I-i}\right)^2 \hat{\sigma}_{I-i}^2 \cdot \frac{1}{C_{i,I-i} \prod_{l=I-i}^{I-1} \bar{f}_l} \\
&\approx \sum_{i=1}^{I} \left(\hat{U}_{i,I-i}\right)^2 \left(\prod_{l=I-i}^{I-1} \bar{f}_l\right) \cdot \frac{\hat{s}_{I-i}^2}{(\bar{f}_{I-i})^2} \cdot \frac{1}{C_{i,I-i} \prod_{l=I-i}^{I-1} \bar{f}_l} \\
&= \sum_{i=1}^{I} \left(\hat{U}_{i,I-i}\right)^2 \cdot \frac{\hat{s}_{I-i}^2}{(\bar{f}_{I-i})^2} \cdot \left[\frac{1}{C_{i,I-i}}\right],
\end{aligned}$$

where in the above computation I did use the approximation

$$\begin{aligned}
\hat{\sigma}_j^2 &\overset{(6.5)}{\approx} \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} \hat{U}_{i,j} \left(\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}} - 1\right)^2 \\
&\overset{1)}{=} \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} \hat{U}_{i,j} \left(\frac{C_{i,j+1} \cdot \prod_{l=j+1}^{I-1} \bar{f}_l}{C_{i,j} \cdot \prod_{l=j}^{I-1} \bar{f}_l} - 1\right)^2 \\
&= \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} C_{i,j} \cdot \prod_{l=j}^{I-1} \bar{f}_l \left(\frac{C_{i,j+1}}{C_{i,j} \cdot \bar{f}_j} - 1\right)^2 \\
&= \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} C_{i,j} \cdot \prod_{l=j}^{I-1} \bar{f}_l \cdot \frac{1}{(\bar{f}_j)^2} \left(\frac{C_{i,j+1}}{C_{i,j}} - \bar{f}_j\right)^2 \\
&= \left(\frac{1}{(\bar{f}_j)^2} \cdot \prod_{l=j}^{I-1} \bar{f}_l\right) \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} C_{i,j} \left(\frac{C_{i,j+1}}{C_{i,j}} - \bar{f}_j\right)^2 \\
&\overset{2)}{\approx} \left(\prod_{l=j}^{I-1} \bar{f}_l\right) \cdot \frac{\hat{s}_j^2}{(\bar{f}_j)^2}, \quad j \in \{0, \ldots, I-2\},
\end{aligned}$$

and the additional assumption 3) i.e., $\hat{\sigma}_{I-1}^2 \approx \bar{f}_{I-1} \cdot \frac{\hat{s}_{I-1}^2}{(\bar{f}_{I-1})^2}$.

Therefore the first approximation is proved, since the Merz-Wüthrich (2008, 2014) formula under no parameter estimation uncertainty reduces to

$$\text{msep}^{\text{MW}}_{\sum_{i=0}^{I} \widehat{\text{CDR}}_i(I+1)|\mathcal{F}_I}(0) = \sum_{i=1}^{I} \left(\hat{U}_{i,I-i}\right)^2 \frac{\hat{s}_{I-i}^2}{\left(\hat{f}_{I-i}^{(I)}\right)^2} \left[\frac{1}{C_{i,I-i}}\right].$$

For the total run-off risk, it holds true that

$$\begin{aligned}
&\widehat{\text{msep}}_{\sum_{i=0}^{I} U^i|\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I} U^i \middle| \mathcal{F}_I\right]\right) \\
&\overset{(6.5)}{\approx} \sum_{i=1}^{I} \left[\sum_{j=I-i}^{I-1} \hat{\sigma}_j^2\right] \cdot \hat{U}_{i,I-i} \\
&\overset{1)}{=} \sum_{i=1}^{I} \left(\hat{U}_{i,I-i}\right)^2 \cdot \sum_{j=I-i}^{I-1} \hat{\sigma}_j^2 \cdot \left[\frac{1}{C_{i,I-i} \prod_{l=I-i}^{I-1} \bar{f}_l}\right] \\
&\approx \sum_{i=1}^{I} \left(\hat{U}_{i,I-i}\right)^2 \cdot \sum_{j=I-i}^{I-1} \left(\prod_{l=j}^{I-1} \bar{f}_l\right) \cdot \frac{\hat{s}_j^2}{(\bar{f}_j)^2} \cdot \left[\frac{1}{C_{i,I-i} \prod_{l=I-i}^{I-1} \bar{f}_l}\right],
\end{aligned}$$

where in the last step of the above computation I did use again the approximation

$$\hat{\sigma}_j^2 \approx \left(\prod_{l=j}^{I-1} \bar{f}_l\right) \cdot \frac{\hat{s}_j^2}{(\bar{f}_j)^2}, \quad j \in \{0, \ldots, I-1\}.$$

Therefore, the second approximation is proved since Mack's (1993) formula under no parameter estimation uncertainty reduces to

$$\text{msep}^{\text{Mack}}_{\sum_{i=0}^{I} U^i | \mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I} U^i \middle| \mathcal{F}_I\right]\right)$$

$$= \sum_{i=1}^{I}\left(\hat{U}_{i,I-i}\right)^2 \sum_{j=I-i}^{I-1} \frac{\hat{s}_j^2}{\left(\hat{f}_j^{(I)}\right)^2}\left[\frac{1}{C_{i,I-i} \cdot \prod_{l=I-i}^{j-1} \hat{f}_l^{(I)}}\right]. \qquad \square$$

Remarks:

- The assumptions I have taken to be able to prove the above-stated approximations are rather restrictive.
- In the above proof, due to the fact that Mack's (1993) distribution-free CL model does fulfill the unconditional unbiased property, I should have done the comparison even better using the parameter estimators given by

$$\hat{g}_j = 1, \quad j \in \{0, \ldots, I-1\},$$

$$\hat{\sigma}_j^2 = \frac{1}{I-j} \sum_{i=0}^{I-j-1} \hat{U}_{i,j}\left(\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}} - 1\right)^2, \quad j \in \{0, \ldots, I-2\}.$$

In that respect, note that the stated approximations would be well fulfilled anyway.

In the next section, for didactic purposes based on a toy numerical example, I will relax assumptions 1) and 3), i.e., only assume the stability conditions $\hat{f}_j^{(k)} \approx \bar{f}_j$, $\forall j, k$ to hold true, and compare again the result obtained by applying my formulas with the result obtained by applying Mack's (1993) and Merz and Wüthrich's (2008) formulas.

6.2. Toy numerical example

I will consider the claims payments data $(C_{i,j})$ shown in Table 12.

Table 12. Claims payments $(C_{i,j})$: Toy example

i/j	0	1	2	3	4
0	2,357	7,432	12,444	16,639	16,738
1	8,345	26,046	43,651	56,832	
2	5,492	16,799	26,999		
3	7,688	23,695			
4	4,566				

The related estimated CL factors $(\hat{f}_j^{(4)})$ can be evaluated to be equal to the values

j	0	1	2	3
$\hat{f}_j^{(4)}$	3.097	1.653	1.310	1.006

and the CL variance parameter estimators

$$\hat{s}_j^2 := \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} C_{i,j}\left(\frac{C_{i,j+1}}{C_{i,j}} - \hat{f}_j^{(4)}\right)^2,$$

$$j \in \{0, \ldots, I-2\}$$

$$\hat{s}_{I-1}^2 := \min\left\{\hat{s}_{I-3}^2, s_{I-2}^2, \frac{\hat{s}_{I-2}^4}{\hat{s}_{I-3}^2}\right\}$$

can be evaluated to be equal to the values

j	0	1	2	3
$\hat{s}_j^2$	7.340	26.173	11.962	5.467

In the next subsection, I will consider three choices of a priori expected CL factors and compute for each choice the prediction uncertainties.

6.2.1. Choice 1

I choose the a priori expected factors $(\bar{f}_j)$ equal to the estimated CL factors $(\hat{f}_j^{(4)})$. Then I obtain the series of estimated CL factors shown in Table 13.

The related triangle of ultimate estimates $(\hat{U}_{i,j})$ is given in Table 14.

According to estimators (3.1), (3.2) and (3.3) the estimated parameters $(\hat{g}_j)$ and $(\hat{\sigma}_j^2)$ in my model are therefore given in Table 15.

Table 13. Series of estimated CL factors: Toy example, first choice of the a priori expected factors $(\bar{f}_j)$

j	0	1	2	3
$\bar{f}_j$	3.097	1.653	1.310	1.006
$\hat{f}_j^{(0)}$	3.097	1.653	1.310	1.006
$\hat{f}_j^{(1)}$	3.153	1.653	1.310	1.006
$f_j^{(2)}$	3.128	1.674	1.310	1.006
$\hat{f}_j^{(3)}$	3.105	1.676	1.337	1.006
$\hat{f}_j^{(4)}$	3.097	1.653	1.310	1.006

Table 14. Ultimate estimates ($\hat{U}_{i,j}$): Toy example, first choice of the a priori expected factors ($\bar{f}_j$)

i/j	0	1	2	3	4
0	15,897	16,184	16,396	16,738	16,738
1	57,298	57,460	58,713	57,170	
2	37,901	37,861	35,573		
3	53,794	51,597			
4	30,796				

Table 15. Estimated parameters ($\hat{g}_j$) and ($\hat{\sigma}_j^2$): Toy example, first choice of the a priori expected factors ($\bar{f}_j$)

j	0	1	2	3
$\hat{g}_j$	0.9891	0.9926	0.9840	1.0000
$\hat{\sigma}_j^2$	25.3279	81.1887	28.5140	10.0143

Table 16. One-year and total run-off prediction uncertainties: Toy example, first choice of the a priori expected factors ($\bar{f}_j$)

	One-year risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}\widehat{\text{CDR}}_i(I+1)\mid\mathcal{F}_I}(0)$	Total run-off risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}U^i\mid\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I}U^i\mid\mathcal{F}_I\right]\right)$
My model	2,864	4,490
Underlying CL model	3,629	4,114

Finally, the estimated prediction uncertainties can be computed to the results shown in Table 16.

This choice of the a priori expected factors is rather artificial. In the next subsection, I will do a more realistic choice.

6.2.2. Choice 2

I choose the a priori expected factors ($\bar{f}_j$) to be in line with the estimated CL factors ($\hat{f}_j^{(4)}$). Then I obtain the series of estimated CL factors shown in Table 17.

The related triangle of ultimate estimates ($\hat{U}_{i,j}$) is given in Table 18.

Table 17. Series of estimated CL factors: Toy example, second choice of the a priori expected factors ($\bar{f}_j$)

j	0	1	2	3
$\bar{f}_j$	3.100	1.700	1.300	1.010
$\hat{f}_j^{(0)}$	3.100	1.700	1.300	1.010
$\hat{f}_j^{(1)}$	3.153	1.700	1.300	1.010
$\hat{f}_j^{(2)}$	3.128	1.674	1.300	1.010
$\hat{f}_j^{(3)}$	3.105	1.676	1.337	1.010
$\hat{f}_j^{(4)}$	3.097	1.653	1.310	1.006

Table 18. Ultimate estimates ($\hat{U}_{i,j}$): Toy example, second choice of the a priori expected factors ($\bar{f}_j$)

i/j	0	1	2	3	4
0	16,309	16,589	16,339	16,805	16,738
1	58,734	57,261	58,950	57,170	
2	37,770	38,013	35,573		
3	54,011	51,597			
4	30,796				

Table 19. Estimated parameters ($\hat{g}_j$) and ($\hat{\sigma}_j^2$): Toy example, second choice of the a priori expected factors ($\bar{f}_j$)

j	0	1	2	3
$\hat{g}_j$	0.9798	0.9910	0.9826	0.9960
$\hat{\sigma}_j^2$	27.7894	100.6560	44.1363	19.3532

Table 20. One-year and total run-off prediction uncertainties: Toy example, second choice of the a priori expected factors ($\bar{f}_j$)

	One-year risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}\widehat{\text{CDR}}_i(I+1)\mid\mathcal{F}_I}(0)$	Total run-off risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}U^i\mid\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I}U^i\mid\mathcal{F}_I\right]\right)$
My model	3,530	5,808
Underlying CL model	3,629	4,114

The estimated parameters ($\hat{g}_j$) and ($\hat{\sigma}_j^2$) in my model are therefore given in Table 19.

Finally, the estimated prediction uncertainties can be computed to the results shown in Table 20.

6.2.3. Choice 3

I choose the a priori expected factors ($\bar{f}_j$) to be less similar to the estimated CL factors ($\hat{f}_j^{(4)}$) than in choice 2. Then I obtain the following series of estimated CL factors in Table 21.

The related triangle of ultimate estimates ($\hat{U}_{i,j}$) is given in Table 22.

Table 21. Series of estimated CL factors: Toy example, third choice of the a priori expected factors ($\bar{f}_j$)

j	0	1	2	3
$\bar{f}_j$	3.000	1.750	1.250	1.000
$\hat{f}_j^{(0)}$	3.000	1.750	1.250	1.000
$\hat{f}_j^{(1)}$	3.153	1.750	1.250	1.000
$\hat{f}_j^{(2)}$	3.128	1.674	1.250	1.000
$\hat{f}_j^{(3)}$	3.105	1.676	1.337	1.000
$\hat{f}_j^{(4)}$	3.097	1.653	1.310	1.006

Table 22. Ultimate estimates ($\hat{U}_{i,j}$): Toy example, third choice of the a priori expected factors ($\bar{f}_j$)

i/j	0	1	2	3	4
0	15,468	16,258	15,555	16,639	16,738
1	57,560	54,514	58,366	57,170	
2	35,957	37,637	35,573		
3	53,476	51,597			
4	30,796				

Table 23. Estimated parameters ($\hat{g}_j$) and ($\hat{\sigma}_j^2$): Toy example, third choice of the a priori expected factors ($\bar{f}_j$)

j	0	1	2	3
$\hat{g}_j$	0.9849	1.0100	0.9985	1.0059
$\hat{\sigma}_j^2$	102.9613	202.4903	99.8821	49.2687

Table 24. One-year and total run-off prediction uncertainties: Toy example, third choice of the a priori expected factors ($\bar{f}_j$)

	One-year risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}\widehat{\text{CDR}}_i(I+1)\|\mathcal{F}_I}(0)$	Total run-off risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}U^i\|\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I}U^i\|\mathcal{F}_I\right]\right)$
My model	4,484	6,487
Underlying CL model	3,629	4,114

The estimated parameters ($\hat{g}_j$) and ($\hat{\sigma}_j^2$) in my model are therefore given in Table 23.

Finally, the estimated prediction uncertainties can be computed to the results shown in Table 24.

Looking at the above results, I observe that if the a priori expected CL factors do not differ too much from the estimated CL factors, then the magnitude order of the estimated prediction uncertainties under the two model assumptions is similar.

Moreover, since within Mack's (1993) model the unconditional unbiased property is fulfilled, for comparison purposes I should better evaluate my formulas using the modified parameter estimators as explained in Section 3.1. When doing this, I note that the results are even more aligned, as shown in Table 25.

Remark:

Coming back to my main numerical example, I have to note that the ultimate estimates triangle considered has been derived starting from the claims payments data ($C_{i,j}$) given in Table 5 by applying, constantly over time, the standard CL methodology, i.e., using the series of estimated CL factors shown in Table 26.

Therefore, based on the claims payments triangle I can evaluate both the Mack (1993) and the Merz-Wüthrich (2008) formulas and compare them with my results in Table 27.

7. Conclusion

In actuarial practice, a large number of reserving methodologies are applied. Unfortunately not all these reserving methodologies are supported by a fully well-defined stochastic reserving model, and for some methodologies that are supported by such a stochastic model, the reserve risk uncertainties estimators may be unknown.

In all of these cases, it is currently not possible to properly estimate the reserve risk uncertainties.

In my paper, I did provide a solution to overcome this problem since I derived estimators that depend on only the historical triangle of ultimate estimates and can therefore be evaluated even if the reserving methodology is, from a pure stochastic point of view, not fully well defined.

I therefore believe that my formulas could be particularly useful to estimate the reserve risk uncertainties

Table 25. Comparison of prediction uncertainties: Toy example, third choice of the a priori expected factors ($\bar{f}_j$)

	One-year risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}\widehat{\text{CDR}}_i(I+1)\|\mathcal{F}_I}(0)$	Total run-off risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}U^i\|\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I}U^i\|\mathcal{F}_I\right]\right)$
My model	4,484	6,487
My model with ($\hat{g}_j = 1$) and adjusted ($\hat{\sigma}_j^2$)	3,554	4,811
Underlying CL model	3,629	4,114

Table 26. Series of estimated CL factors: Numerical example

j	0	1	2	3	4	5	6	7	8	9	10	11
$\bar{f}_j$	1.4000	1.0400	1.0100	1.0050	1.0050	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{f}_j^{(0)}$	1.4000	1.0400	1.0100	1.0050	1.0050	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{f}_j^{(1)}$	1.4000	1.0400	1.0100	1.0050	1.0050	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{f}_j^{(2)}$	1.4000	1.0325	1.0100	1.0050	1.0050	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{f}_j^{(3)}$	1.4000	1.0299	1.0059	1.0050	1.0050	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{f}_j^{(4)}$	1.4000	1.0275	1.0058	1.0014	1.0050	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{f}_j^{(5)}$	1.3949	1.0264	1.0121	1.0058	1.0111	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{f}_j^{(6)}$	1.4036	1.0255	1.0111	1.0091	1.0082	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{f}_j^{(7)}$	1.4086	1.0246	1.0106	1.0075	1.0059	1.0001	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000
$\hat{f}_j^{(8)}$	1.4190	1.0245	1.0093	1.0078	1.0044	1.0001	1.0001	0.9996	1.0000	1.0000	1.0000	1.0000
$\hat{f}_j^{(9)}$	1.4248	1.0241	1.0094	1.0078	1.0061	1.0000	0.9998	0.9997	0.9997	1.0000	1.0000	1.0000
$\hat{f}_j^{(10)}$	1.4339	1.0245	1.0091	1.0067	1.0065	1.0003	0.9998	0.9998	1.0000	0.9998	1.0000	1.0000
$\hat{f}_j^{(11)}$	1.4366	1.0312	1.0117	1.0062	1.0056	1.0003	1.0003	0.9998	1.0002	0.9999	1.0000	1.0000
$\hat{f}_j^{(12)}$	1.4375	1.0309	1.0119	1.0060	1.0049	1.0003	1.0003	0.9998	1.0001	0.9999	1.0000	1.0000

(both one-year and total run-off) of a given insurance portfolio in case

- the prediction uncertainties related to the stochastic model supporting the applied reserving methodology are not yet known; or
- we have a probabilistic lack of consistency in the stochastic model supporting the applied reserving methodology that could be due, for instance (as is often usual in practice), to
  - different basic reserving methodologies applied to different accident years (e.g., BF methodology for less-mature accident years and CL methodology for more-mature accident years) or
  - mixtures of basic methodologies applied to specific accident years; or
- we do not know exactly according to which reserving methodology the ultimate estimates have been generated (e.g., if the reserving analysis is done on a more granular level, then the reserve risk uncertainties need to be quantified for solvency purposes).

Finally, I also believe that if, for a given insurance portfolio, a well-defined reserving methodology is applied (for which a fully consistent and validated supporting model exists and the prediction uncertainties estimators within that model are known and fulfill good properties), then the corresponding estimators (without forgetting to allow also for a model risk loading) should be preferred to those presented in this paper.

Nevertheless, based on numerical examples, I did show that in the standard CL framework and under reasonable stability conditions, my high-level formulas deliver results similar to the Mack (1993) and Merz-Wüthrich (2008) formulas.

Table 27. Comparison of prediction uncertainties: Numerical example

	One-year risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}\widehat{\text{CDR}}_i(I+1)\mid\mathcal{F}_I}(0)$	Total run-off risk $\widehat{\text{msep}}^{1/2}_{\sum_{i=0}^{I}U^i\mid\mathcal{F}_I}\left(\hat{E}\left[\sum_{i=0}^{I}U^i\mid\mathcal{F}_I\right]\right)$
My model	12,025	15,228
My model with ($\hat{g}_j = 1$) and adjusted ($\hat{\sigma}_j^2$)	11,080	13,687
Underlying CL model	11,203 (Merz-Wüthrich)	13,457 (Mack)

Appendices

Appendix A.1. Unbiasedness of the estimators $\hat{g}_j$ and $\hat{\sigma}_j^2$

In this section, I prove in detail the conditional unbiasedness of the estimators $\hat{g}_j$ and $\hat{\sigma}_j^2$. It holds true that

$$
\begin{aligned}
E\left[\hat{g}_j \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] &= E\left[\frac{\sum_{i=0}^{I-j-1}\hat{U}_{i,j+1}}{\sum_{i=0}^{I-j-1}\hat{U}_{i,j}} \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] \\
&= \frac{1}{\sum_{i=0}^{I-j-1}\hat{U}_{i,j}} E\left[\sum_{i=0}^{I-j-1}\hat{U}_{i,j+1} \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] \\
&= \frac{1}{\sum_{i=0}^{I-j-1}\hat{U}_{i,j}} \sum_{i=0}^{I-j-1} E\left[E\left[\hat{U}_{i,j+1} \middle| \mathcal{D}_j^{j+i}\right] \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] \\
&= \frac{1}{\sum_{i=0}^{I-j-1}\hat{U}_{i,j}} \sum_{i=0}^{I-j-1} E\left[g_j \cdot \hat{U}_{i,j} \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] = g_j, \quad j \in \{0,\ldots,I-1\},
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\hat{\sigma}_j^2 \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] \\
&= E\left[\frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\hat{U}_{i,j}\left(\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}}-\hat{g}_j\right)^2 \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] \\
&= \frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\hat{U}_{i,j}E\left[\left(\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}}-g_j\right)^2+\left(\hat{g}_j-g_j\right)^2-2\left(\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}}-g_j\right)\left(\hat{g}_j-g_j\right) \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] \\
&= \frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\hat{U}_{i,j}Var\left(\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}} \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right)+\frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\hat{U}_{i,j}E\left[\left(\hat{g}_j-g_j\right)^2 \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] \\
&\quad -\frac{1}{I-j-1}E\left[2\sum_{i=0}^{I-j-1}\hat{U}_{i,j}\left(\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}}-g_j\right)\left(\hat{g}_j-g_j\right) \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] \\
&= \frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\hat{U}_{i,j}\cdot\frac{\sigma_j^2}{\hat{U}_{i,j}}+\frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\hat{U}_{i,j}E\left[\left(\hat{g}_j-g_j\right)^2 \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] \\
&\quad -\frac{1}{I-j-1}\left(\sum_{i=0}^{I-j-1}\hat{U}_{i,j}\right)\cdot E\left[2\left(\frac{\sum_{i=0}^{I-j-1}\hat{U}_{i,j+1}}{\sum_{i=0}^{I-j-1}\hat{U}_{i,j}}-g_j\right)\left(\hat{g}_j-g_j\right) \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] \\
&= \frac{1}{I-j-1}(I-j)\sigma_j^2-\frac{1}{I-j-1}\left(\sum_{i=0}^{I-j-1}\hat{U}_{i,j}\right)\cdot E\left[\left(\hat{g}_j-g_j\right)^2 \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right] \\
&= \frac{1}{I-j-1}(I-j)\sigma_j^2-\frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\hat{U}_{i,j}\cdot Var\left(\hat{g}_j \middle| \sigma\left(\bigcup_{i=0}^{I}\left\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\right\}\right)\right)
\end{aligned}
$$

$$= \frac{1}{I-j-1}(I-j)\sigma_j^2 - \frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\hat{U}_{i,j}\cdot Var\left(\sum_{i=0}^{I-j-1}\frac{\hat{U}_{i,j}}{\sum_{i=0}^{I-j-1}\hat{U}_{i,j}}\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}}\middle|\sigma\left(\bigcup_{i=0}^{I}\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\}\right)\right)$$

$$= \frac{1}{I-j-1}(I-j)\sigma_j^2 - \frac{1}{I-j-1}\sum_{i=0}^{I-j-1}\hat{U}_{i,j}\cdot\sum_{i=0}^{I-j-1}\frac{\hat{U}_{i,j}^2}{\left(\sum_{i=0}^{I-j-1}\hat{U}_{i,j}\right)^2}Var\left(\frac{\hat{U}_{i,j+1}}{\hat{U}_{i,j}}\middle|\sigma\left(\bigcup_{i=0}^{I}\{\hat{U}_{i,0},\ldots,\hat{U}_{i,j}\}\right)\right)$$

$$= \frac{1}{I-j-1}(I-j)\sigma_j^2 - \frac{1}{I-j-1}\sigma_j^2 = \sigma_j^2, \quad j\in\{0,\ldots,I-2\}.$$

Appendix B.1. One-year uncertainty estimator for single accident year

I estimate the one-year uncertainty for single accident year $E[(\hat{U}_{i,I-i+1} - \hat{U}_{i,I-i})^2|\mathcal{F}_I]$ by

$$\hat{E}\left[\left(\hat{U}_{i,I-i+1}-\hat{U}_{i,I-i}\right)^2\middle|\mathcal{D}^I\right] = \hat{E}\left[\hat{U}_{i,I-i+1}^2\middle|\mathcal{D}^I\right] - 2\hat{E}\left[\hat{U}_{i,I-i+1}\hat{U}_{i,I-i}\middle|\mathcal{D}^I\right] + \hat{E}\left[\hat{U}_{i,I-i}^2\middle|\mathcal{D}^I\right]$$

$$= \left(\widehat{Var}\left(\hat{U}_{i,I-i+1}\middle|\mathcal{D}^I\right) + \hat{E}\left[\hat{U}_{i,I-i+1}\middle|\mathcal{D}^I\right]^2\right) - 2\left[\begin{array}{l}\hat{E}\left[\hat{U}_{i,I-i+1}\middle|\mathcal{D}^I\right]\hat{E}\left[\hat{U}_{i,I-i}\middle|\mathcal{D}^I\right]\\ +\widehat{Cov}\left(\hat{U}_{i,I-i+1},\hat{U}_{i,I-i}\middle|\mathcal{D}^I\right)\end{array}\right]$$

$$+\left(\underbrace{\widehat{Var}\left(\hat{U}_{i,I-i}\middle|\mathcal{D}^I\right)}_{=0} + \hat{E}\left[\hat{U}_{i,I-i}\middle|\mathcal{D}^I\right]^2\right)$$

$$= \left(\hat{\sigma}_{I-i}^2\cdot\hat{U}_{i,I-i} + \hat{U}_{i,I-i}^2\cdot\hat{g}_{I-i}^2\right) - 2\cdot\left[\hat{U}_{i,I-i}^2\cdot\hat{g}_{I-i} + \underbrace{\widehat{Cov}\left(\hat{U}_{i,I-i+1},\hat{U}_{i,I-i}\middle|\mathcal{D}^I\right)}_{=0}\right] + \hat{U}_{i,I-i}^2$$

$$= \hat{\sigma}_{I-i}^2\cdot\hat{U}_{i,I-i} + (\hat{g}_{I-i}-1)^2\cdot\hat{U}_{i,I-i}^2, \quad i\geq 1.$$

Appendix B.2. Split of the one-year prediction uncertainty

For a single accident year $i \geq 1$, MSEP of the one-year prediction uncertainty can be split into three components (process variance, parameter&model uncertainty, and mixed term) as follows:

$$\mathrm{msep}_{\widehat{\mathrm{CDR}}_i(I+1)|\mathcal{F}_I}(0) := E\left[\left(\hat{E}[U^i|\mathcal{F}_{I+1}] - \hat{E}[U^i|\mathcal{F}_I] - 0\right)^2\middle|\mathcal{F}_I\right]$$

$$= E\left[\left(\hat{E}[U^i|\mathcal{F}_{I+1}] - E[U^i|\mathcal{F}_{I+1}] + E[U^i|\mathcal{F}_I] - \hat{E}[U^i|\mathcal{F}_I] + E[U^i|\mathcal{F}_{I+1}] - E[U^i|\mathcal{F}_I]\right)^2\middle|\mathcal{F}_I\right]$$

$$= E\left[\left(E[U^i|\mathcal{F}_{I+1}] - E[E[U^i|\mathcal{F}_{I+1}]|\mathcal{F}_I]\right)^2\middle|\mathcal{F}_I\right] + E\left[\left(\hat{U}_{i,I-i+1} - E[U^i|\mathcal{F}_{I+1}]\right)^2\middle|\mathcal{F}_I\right]$$

$$+ E\left[\left(E[U^i|\mathcal{F}_I] - \hat{U}_{i,I-i}\right)^2\middle|\mathcal{F}_I\right] + 2\left(E[U^i|\mathcal{F}_I] - \hat{U}_{i,I-i}\right)E\left[\left(\hat{U}_{i,I-i+1} - E[U^i|\mathcal{F}_{I+1}]\right)\middle|\mathcal{F}_I\right]$$

$$+ 2\left(E[U^i|\mathcal{F}_I] - \hat{U}_{i,I-i}\right)\underbrace{E\left[\left(E[U^i|\mathcal{F}_{I+1}] - E[U^i|\mathcal{F}_I]\right)\middle|\mathcal{F}_I\right]}_{=0}$$

$$+ 2E\left[\left(\hat{U}_{i,I-i+1} - E[U^i|\mathcal{F}_{I+1}]\right)\left(E[U^i|\mathcal{F}_{I+1}] - E[U^i|\mathcal{F}_I]\right)\middle|\mathcal{F}_I\right]$$

$$= \underbrace{Var\left(E[U^i|\mathcal{F}_{I+1}]|\mathcal{F}_I\right)}_{\text{process variance}} + \underbrace{E\left[\left(\hat{U}_{i,I-i+1} - E[U^i|\mathcal{F}_{I+1}]\right)^2 \Big| \mathcal{F}_I\right] + E\left[\left(\hat{U}_{i,I-i} - E[U^i|\mathcal{F}_I]\right)^2 \Big| \mathcal{F}_I\right]}_{\text{parameter\&model uncertainty: first term}}$$

$$- \underbrace{2E\left[\left(\hat{U}_{i,I-i} - E[U^i|\mathcal{F}_I]\right)\left(\hat{U}_{i,I-i+1} - E[U^i|\mathcal{F}_{I+1}]\right)\Big|\mathcal{F}_I\right]}_{\text{parameter\&model uncertainty: second term}}$$

$$+ \underbrace{2E\left[\left(\hat{U}_{i,I-i+1} - E[U^i|\mathcal{F}_{I+1}]\right)\left(E[U^i|\mathcal{F}_{I+1}] - E[U^i|\mathcal{F}_I]\right)\Big|\mathcal{F}_I\right]}_{\text{mixed term}}.$$

Appendix B.3. One-year process variance estimator

I estimate the process variance $Var(E[U^i|\mathcal{F}_{I+1}]|\mathcal{F}_I)$ by

$$\begin{aligned}
\widehat{Var}\left(E[U^i|\mathcal{D}^{I+1}]|\mathcal{D}^I\right) &= \widehat{Var}\left(E\left[\hat{U}_{i,I}|\mathcal{D}^{I+1}\right]|\mathcal{D}^I\right) \\
&= \widehat{Var}\left(\hat{U}_{i,I-i+1}\prod_{j=I-i+1}^{I-1} g_j \Big| \mathcal{D}^I\right) \\
&= \left(\prod_{j=I-i+1}^{I-1} \hat{g}_j\right)^2 \cdot \widehat{Var}\left(\hat{U}_{i,I-i+1}|\mathcal{D}^I\right) \\
&= \left(\prod_{j=I-i+1}^{I-1} \hat{g}_j\right)^2 \cdot \hat{\sigma}^2_{I-i} \cdot \hat{U}_{i,I-i}, \quad i \geq 1.
\end{aligned}$$

Appendix B.4. One-year parameter&model uncertainty estimator

I estimate the first term of the parameter&model uncertainty

$$E\left[\left(\hat{U}_{i,I-i+1} - E[U^i|\mathcal{F}_{I+1}]\right)^2\Big|\mathcal{F}_I\right] + E\left[\left(\hat{U}_{i,I-i} - E[U^i|\mathcal{F}_I]\right)^2\Big|\mathcal{F}_I\right]$$

by

$$\begin{aligned}
&\hat{E}\left[\left(\hat{U}_{i,I-i+1} - E[U^i|\mathcal{D}^{I+1}]\right)^2\Big|\mathcal{D}^I\right] + \hat{E}\left[\left(\hat{U}_{i,I-i} - E[U^i|\mathcal{D}^I]\right)^2\Big|\mathcal{D}^I\right] \\
&= \hat{E}\left[\left(\hat{U}_{i,I-i+1} - \hat{U}_{i,I-i+1}\prod_{j=I-i+1}^{I-1} g_j\right)^2\Bigg|\mathcal{D}^I\right] + \hat{E}\left[\left(\hat{U}_{i,I-i}\prod_{j=I-i}^{I-1} g_j - \hat{U}_{i,I-i}\right)^2\Bigg|\mathcal{D}^I\right] \\
&= \left(1 - \prod_{j=I-i+1}^{I-1}\hat{g}_j\right)^2 \cdot \hat{E}\left[\hat{U}^2_{i,I-i+1}|\mathcal{D}^I\right] + \left(\prod_{j=I-i}^{I-1}\hat{g}_j - 1\right)^2 \cdot \hat{U}^2_{i,I-i} \\
&= \left(1 - \prod_{j=I-i+1}^{I-1}\hat{g}_j\right)^2 \cdot \left(\widehat{Var}\left(\hat{U}_{i,I-i+1}|\mathcal{D}^I\right) + \hat{E}\left[\hat{U}_{i,I-i+1}|\mathcal{D}^I\right]^2\right) + \left(\prod_{j=I-i}^{I-1}\hat{g}_j - 1\right)^2 \cdot \hat{U}^2_{i,I-i} \\
&= \left(1 - \prod_{j=I-i+1}^{I-1}\hat{g}_j\right)^2 \cdot \left(\hat{\sigma}^2_{I-i} \cdot \hat{U}_{i,I-i} + \hat{g}^2_{I-i} \cdot \hat{U}^2_{i,I-i}\right) + \left(\prod_{j=I-i}^{I-1}\hat{g}_j - 1\right)^2 \cdot \hat{U}^2_{i,I-i} \\
&= \left(1 - \prod_{j=I-i+1}^{I-1}\hat{g}_j\right)^2 \cdot \hat{\sigma}^2_{I-i} \cdot \hat{U}_{i,I-i} + \left(\hat{g}_{I-i} - \prod_{j=I-i}^{I-1}\hat{g}_j\right)^2 \cdot \hat{U}^2_{i,I-i} + \left(1 - \prod_{j=I-i}^{I-1}\hat{g}_j\right)^2 \cdot \hat{U}^2_{i,I-i}, \quad i \geq 1.
\end{aligned}$$

I estimate the second term of the parameter&model uncertainty

$$-2E\left[\left(\hat{U}_{i,I-i}-E[U^i|\mathcal{F}_I]\right)\left(\hat{U}_{i,I-i+1}-E[U^i|\mathcal{F}_{I+1}]\right)\middle|\mathcal{F}_I\right]$$

by

$$\begin{aligned}
&-2\hat{E}\left[\left(\hat{U}_{i,I-i}-E[U^i|\mathcal{D}^I]\right)\left(\hat{U}_{i,I-i+1}-E[U^i|\mathcal{D}^{I+1}]\right)\middle|\mathcal{D}^I\right]\\
&=-2\hat{E}\left[\left(\hat{U}_{i,I-i}-\hat{U}_{i,I-i}\prod_{j=I-i}^{I-1}g_j\right)\left(\hat{U}_{i,I-i+1}-\hat{U}_{i,I-i+1}\prod_{j=I-i+1}^{I-1}g_j\right)\middle|\mathcal{D}^I\right]\\
&=-2\left(1-\prod_{j=I-i}^{I-1}\hat{g}_j\right)\left(1-\prod_{j=I-i+1}^{I-1}\hat{g}_j\right)\cdot\hat{U}_{i,I-i}\cdot\hat{E}\left[\hat{U}_{i,I-i+1}\middle|\mathcal{D}^I\right]\\
&=-2\left(1-\prod_{j=I-i}^{I-1}\hat{g}_j\right)\left(\hat{g}_{I-i}-\prod_{j=I-i}^{I-1}\hat{g}_j\right)\cdot\hat{U}^2_{i,I-i},\quad i\geq 1.
\end{aligned}$$

Appendix B.5 One-year mixed term estimator

I estimate the mixed term $2E[(\hat{U}_{i,I-i+1}-E[U^i|\mathcal{F}_{I+1}])(E[U^i|\mathcal{F}_{I+1}]-E[U^i|\mathcal{F}_I])|\mathcal{F}_I]$ by

$$\begin{aligned}
&2\hat{E}\left[\left(\hat{U}_{i,I-i+1}-E[U^i|\mathcal{D}^{I+1}]\right)\left(E[U^i|\mathcal{D}^{I+1}]-E[U^i|\mathcal{D}^I]\right)\middle|\mathcal{D}^I\right]\\
&=2\hat{E}\left[\left(\hat{U}_{i,I-i+1}-\hat{U}_{i,I-i+1}\prod_{j=I-i+1}^{I-1}g_j\right)\left(\hat{U}_{i,I-i+1}\prod_{j=I-i+1}^{I-1}g_j-\hat{U}_{i,I-i}\prod_{j=I-i}^{I-1}g_j\right)\middle|\mathcal{D}^I\right]\\
&=2\left(1-\prod_{j=I-i+1}^{I-1}\hat{g}_j\right)\cdot\hat{E}\left[\hat{U}_{i,I-i+1}\left(\hat{U}_{i,I-i+1}\prod_{j=I-i+1}^{I-1}g_j-\hat{U}_{i,I-i}\prod_{j=I-i}^{I-1}g_j\right)\middle|\mathcal{D}^I\right]\\
&=2\left(1-\prod_{j=I-i+1}^{I-1}\hat{g}_j\right)\left(\prod_{j=I-i+1}^{I-1}\hat{g}_j\right)\cdot\hat{E}\left[\hat{U}^2_{i,I-i+1}\middle|\mathcal{D}^I\right]-2\left(1-\prod_{j=I-i+1}^{I-1}\hat{g}_j\right)\cdot\hat{U}_{i,I-i}\cdot\left(\prod_{j=I-i}^{I-1}\hat{g}_j\right)\cdot\hat{E}\left[\hat{U}_{i,I-i+1}\middle|\mathcal{D}^I\right]\\
&=2\left(1-\prod_{j=I-i+1}^{I-1}\hat{g}_j\right)\left(\prod_{j=I-i+1}^{I-1}\hat{g}_j\right)\cdot\left(\hat{\sigma}^2_{I-1}\cdot\hat{U}_{i,I-i}+\hat{g}^2_{I-i}\cdot\hat{U}^2_{i,I-i}\right)-2\left(1-\prod_{j=I-i+1}^{I-1}\hat{g}_j\right)\cdot\left(\prod_{j=I-i}^{I-1}\hat{g}_j\right)\cdot\hat{g}_{I-i}\cdot\hat{U}^2_{i,I-i}\\
&=2\left(1-\prod_{j=I-i+1}^{I-1}\hat{g}_j\right)\left(\prod_{j=I-i+1}^{I-1}\hat{g}_j\right)\cdot\hat{\sigma}^2_{I-i}\cdot\hat{U}_{i,I-i},\quad i\geq 1.
\end{aligned}$$

Appendix B.6. Proof of Lemma 1

Please note that $\hat{U}_{i,I-i}$ is $\mathcal{D}^I$-measurable, and for $i<j$, I also have that $\hat{U}_{j,I-j+1}$ is $\mathcal{D}^I_{I-i}$-measurable. Therefore, for $i<j$, it holds true that

$$\begin{aligned}
Cov\left(\hat{U}_{i,I-i+1},\hat{U}_{i,I-j+1}\middle|\mathcal{D}^I\right)&=E\left[Cov\left(\hat{U}_{i,I-i+1},\underbrace{\hat{U}_{j,I-j+1}}_{\mathcal{D}^I_{I-i}\text{-measurable}}\middle|\mathcal{D}^I_{I-i}\right)\middle|\mathcal{D}^I\right]+Cov\left(E\left[\hat{U}_{i,I-i+1}\middle|\mathcal{D}^I_{I-i}\right],E\left[\hat{U}_{j,I-j+1}\middle|\mathcal{D}^I_{I-i}\right]\middle|\mathcal{D}^I\right)\\
&=E\left[0\middle|\mathcal{D}^I\right]+Cov\left(g_{I-i}\cdot\underbrace{\hat{U}_{i,I-i}}_{\mathcal{D}^I\text{-measurable}},\hat{U}_{j,I-j+1}\middle|\mathcal{D}^I\right)=0+0=0.
\end{aligned}$$

Appendix B.7. Estimator for the one-year covariance term

Using Lemma 1, I estimate the term

$$E\left[\left(\hat{E}\left[U^i \middle| \mathcal{F}_{I+1}\right]-\hat{E}\left[U^i \middle| \mathcal{F}_I\right]\right)\left(\hat{E}\left[U^j \middle| \mathcal{F}_{I+1}\right]-\hat{E}\left[U^j \middle| \mathcal{F}_I\right]\right) \middle| \mathcal{F}_I\right], \quad i<j,$$

by

$$\begin{aligned}
&\hat{E}\left[\left(\hat{U}_{i,I-i+1}-\hat{U}_{i,I-i}\right)\left(\hat{U}_{j,I-j+1}-\hat{U}_{j,I-j}\right) \middle| \mathcal{D}^I\right] \\
&\quad=\hat{E}\left[\hat{U}_{i,I-i+1}\hat{U}_{j,I-j+1} \middle| \mathcal{D}^I\right]-\hat{E}\left[\hat{U}_{i,I-i+1} \middle| \mathcal{D}^I\right]\hat{U}_{j,I-j}-\hat{U}_{i,I-i}\hat{E}\left[\hat{U}_{j,I-j+1} \middle| \mathcal{D}^I\right]+\hat{U}_{i,I-i}\hat{U}_{j,I-j} \\
&\quad=\hat{E}\left[\hat{U}_{i,I-i+1} \middle| \mathcal{D}^I\right]\cdot\hat{E}\left[\hat{U}_{j,I-j+1} \middle| \mathcal{D}^I\right]-\hat{E}\left[\hat{U}_{i,I-i+1} \middle| \mathcal{D}^I\right]\hat{U}_{j,I-j}-\hat{U}_{i,I-i}\hat{E}\left[\hat{U}_{j,I-j+1} \middle| \mathcal{D}^I\right]+\hat{U}_{i,I-i}\hat{U}_{j,I-j} \\
&\quad=\hat{g}_{I-i}\cdot\hat{U}_{i,I-i}\cdot\hat{g}_{I-j}\cdot\hat{U}_{j,I-j}-\hat{g}_{I-i}\cdot\hat{U}_{i,I-i}\hat{U}_{j,I-j}-\hat{U}_{i,I-i}\cdot\hat{g}_{I-j}\cdot\hat{U}_{j,I-j}+\hat{U}_{i,I-i}\hat{U}_{j,I-j} \\
&\quad=\hat{g}_{I-i}\cdot\hat{g}_{I-j}\cdot\hat{U}_{i,I-i}\hat{U}_{j,I-j}-\hat{g}_{I-i}\cdot\hat{U}_{i,I-i}\hat{U}_{j,I-j}-\hat{g}_{I-j}\cdot\hat{U}_{i,I-i}\hat{U}_{j,I-j}+\hat{U}_{i,I-i}\hat{U}_{j,I-j} \\
&\quad=\left(\hat{g}_{I-i}\cdot\hat{g}_{I-j}-\hat{g}_{I-i}-\hat{g}_{I-j}+1\right)\cdot\hat{U}_{i,I-i}\cdot\hat{U}_{j,I-j}, \quad i<j.
\end{aligned}$$

Appendix C.1. Total run-off process variance estimator for single accident year

It holds true that

$$\begin{aligned}
&\hat{E}\left[\left(\hat{U}_{i,I}-E\left[\hat{U}_{i,I} \middle| \mathcal{D}^I\right]\right)^2 \middle| \mathcal{D}^I\right]=\widehat{Var}\left(\hat{U}_{i,I} \middle| \mathcal{D}^I\right) \\
&\quad=\hat{E}\left[Var\left(\hat{U}_{i,I} \middle| \mathcal{D}_{I-1}^{I-1+i}\right) \middle| \mathcal{D}^I\right]+\widehat{Var}\left(E\left[\hat{U}_{i,I} \middle| \mathcal{D}_{I-1}^{I-1+i}\right] \middle| \mathcal{D}^I\right) \\
&\quad=\hat{E}\left[\sigma_{I-1}^2\cdot\hat{U}_{i,I-1} \middle| \mathcal{D}^I\right]+\widehat{Var}\left(g_{I-1}\cdot\hat{U}_{i,I-1} \middle| \mathcal{D}^I\right) \\
&\quad=\hat{\sigma}_{I-1}^2\cdot\hat{U}_{i,I-i}\cdot\prod_{j=I-i}^{I-2}\hat{g}_j+\hat{g}_{I-1}^2\cdot\widehat{Var}\left(\hat{U}_{i,I-1} \middle| \mathcal{D}^I\right) \\
&\quad=\hat{\sigma}_{I-1}^2\cdot\hat{U}_{i,I-i}\cdot\prod_{j=I-i}^{I-2}\hat{g}_j+\hat{g}_{I-1}^2\cdot\left(\hat{\sigma}_{I-2}^2\cdot\hat{U}_{i,I-i}\cdot\prod_{j=I-i}^{I-3}\hat{g}_j+\hat{g}_{I-2}^2\cdot\widehat{Var}\left(\hat{U}_{i,I-2} \middle| \mathcal{D}^I\right)\right) \\
&\quad=\hat{\sigma}_{I-1}^2\cdot\hat{U}_{i,I-i}\cdot\prod_{j=I-i}^{I-2}\hat{g}_j+\hat{g}_{I-1}^2\cdot\hat{\sigma}_{I-2}^2\cdot\hat{U}_{i,I-i}\cdot\prod_{j=I-i}^{I-3}\hat{g}_j+\hat{g}_{I-1}^2\cdot\hat{g}_{I-2}^2\cdot\widehat{Var}\left(\hat{U}_{i,I-2} \middle| \mathcal{D}^I\right) \\
&\quad=\cdots=\left[\sum_{k=I-i}^{I-1}\left(\prod_{j=I-i}^{k-1}\hat{g}_j\right)\cdot\hat{\sigma}_k^2\cdot\left(\prod_{j=k-1}^{I-1}\hat{g}_j^2\right)\right]\cdot\hat{U}_{i,I-i}, \quad i\geq 1,
\end{aligned}$$

where the last step follows by iteration of the same procedure applied in the prior steps until I reach the triangle diagonal.

Appendix C.2. Total run-off parameter&model uncertainty estimator for single accident year

It holds true that

$$\hat{E}\left[\left(\hat{U}_{i,I-i}-E\left[\hat{U}_{i,I}\middle|\mathcal{D}^I\right]\right)^2\middle|\mathcal{D}^I\right]=\hat{E}\left[\left(\hat{U}_{i,I-i}-\hat{U}_{i,I-i}\prod_{j=I-i}^{I-1}g_j\right)^2\middle|\mathcal{D}^I\right]$$

$$=\hat{E}\left[\hat{U}_{i,I-i}^2\cdot\left(1-\prod_{j=I-i}^{I-1}g_j\right)^2\middle|\mathcal{D}^I\right]$$

$$=\left(1-\prod_{j=I-i}^{I-1}\hat{g}_j\right)^2\cdot\hat{U}_{i,I-i}^2,\quad i\geq 1.$$

Appendix C.3. Proof of Lemma 2

For $i<j$, it holds that $\hat{U}_{i,I}$ is $\mathcal{D}_{I-1}^{I-1+j}$-measurable, and therefore I have the following:

$$Cov\left(\hat{U}_{i,I},\hat{U}_{j,I}\middle|\mathcal{D}^I\right)=E\left[Cov\left(\underbrace{\hat{U}_{i,I}}_{\mathcal{D}_{I-1}^{I-1+j}\text{-measurable}},\hat{U}_{j,I}\middle|\mathcal{D}_{I-1}^{I-1+j}\right)\middle|\mathcal{D}^I\right]+Cov\left(E\left[\hat{U}_{i,I}\middle|\mathcal{D}_{I-1}^{I-1+j}\right],E\left[\hat{U}_{j,I}\middle|\mathcal{D}_{I-1}^{I-1+j}\right]\middle|\mathcal{D}^I\right)$$

$$=E\left[0|\mathcal{D}^I\right]+Cov\left(\hat{U}_{i,I},g_{I-1}\cdot\hat{U}_{j,I-1}\middle|\mathcal{D}^I\right)$$

$$=g_{I-1}\cdot Cov\left(\hat{U}_{i,I},\hat{U}_{j,I-1}\middle|\mathcal{D}^I\right)=\cdots=0,$$

where the last step follows by iteration of the same procedure applied in the prior steps (until I reach the triangle diagonal).

Appendix C.4. Estimator for total run-off covariance term

Using Lemma 2, I estimate the term

$$E\left[\left(\hat{U}_{i,I}-\hat{U}_{i,I-i}\right)\left(\hat{U}_{j,I}-\hat{U}_{j,I-j}\right)\middle|\mathcal{F}_I\right],\quad i<j,$$

by

$$\hat{E}\left[\left(\hat{U}_{i,I}-\hat{U}_{i,I-i}\right)\left(\hat{U}_{j,I}-\hat{U}_{j,I-j}\right)\middle|\mathcal{D}^I\right]$$

$$=\hat{E}\left[\hat{U}_{i,I}\hat{U}_{j,I}\middle|\mathcal{D}^I\right]-\hat{E}\left[\hat{U}_{i,I}\hat{U}_{j,I-j}\middle|\mathcal{D}^I\right]-\hat{E}\left[\hat{U}_{i,I-i}\hat{U}_{j,I}\middle|\mathcal{D}^I\right]+\hat{E}\left[\hat{U}_{i,I-i}\hat{U}_{j,I-j}\middle|\mathcal{D}^I\right]$$

$$=\hat{E}\left[\hat{U}_{i,I}\middle|\mathcal{D}^I\right]\hat{E}\left[\hat{U}_{j,I}\middle|\mathcal{D}^I\right]-\hat{U}_{j,I-j}\cdot\hat{E}\left[\hat{U}_{i,I}\middle|\mathcal{D}^I\right]-\hat{U}_{i,I-i}\cdot\hat{E}\left[\hat{U}_{j,I}\middle|\mathcal{D}^I\right]+\hat{U}_{i,I-i}\hat{U}_{j,I-j}$$

$$=\hat{E}\left[\hat{U}_{i,I}\middle|\mathcal{D}^I\right]\left(\hat{E}\left[\hat{U}_{j,I}\middle|\mathcal{D}^I\right]-\hat{U}_{j,I-j}\right)-\hat{U}_{i,I-i}\cdot\left(\hat{E}\left[\hat{U}_{j,I}\middle|\mathcal{D}^I\right]-\hat{U}_{j,I-j}\right)$$

$$=\left(\hat{U}_{i,I-i}\cdot\prod_{k=I-i}^{I-1}\hat{g}_k-\hat{U}_{i,I-i}\right)\left(\hat{U}_{j,I-j}\cdot\prod_{k=I-j}^{I-1}\hat{g}_k-\hat{U}_{j,I-j}\right)$$

$$=\left(1-\prod_{k=I-i}^{I-1}\hat{g}_k\right)\left(1-\prod_{k=I-j}^{I-1}\hat{g}_k\right)\cdot\hat{U}_{i,I-i}\cdot\hat{U}_{j,I-j},\quad i<j.$$

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